CHM 305 The Quantum World
Lecture 2: Classical Waves

Reading: McQuarrie Chap. 2

Big takeaways from last class?

Road map for today's lecture

- describe the evolution of classical waves in space and time using their ***wavefunction***
- demonstrate how a classical wave obeys a differential equation called the ***wave equation***
- show how the solutions to the wave equation with ***boundary conditions*** give us a set of standing wave solutions
- standing waves will be a nice analogy for solutions to Schrödinger's quantum wave equation

Traveling waves

The ***wavefunction*** of a traveling wave

the “wavefunction” describes how a wave’s amplitude evolves in space and time. Here’s a very simple one:

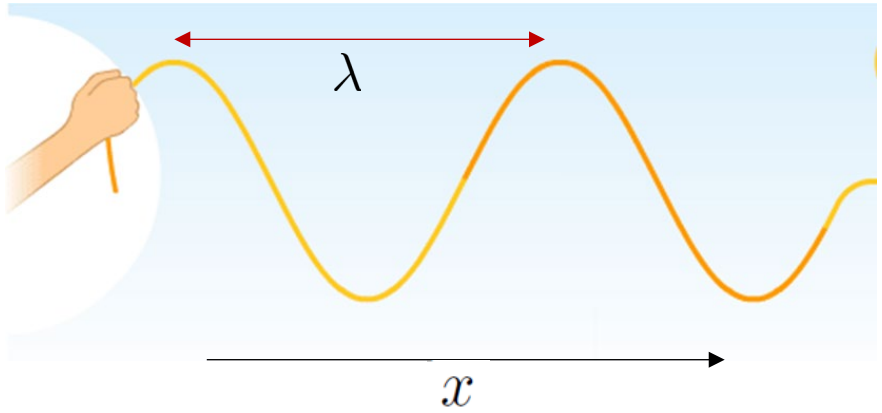
$$\psi(x,t) = A\sin\left(2\pi\left[\frac{x}{\lambda} - \frac{t}{T}\right]\right)$$

amplitude

wavelength

period

$T = 1/\nu$



an equivalent, more compact, notation for this wavefunction:

$$\psi(x,t) = A\sin(kx - \omega t)$$

$k \equiv 2\pi/\lambda$

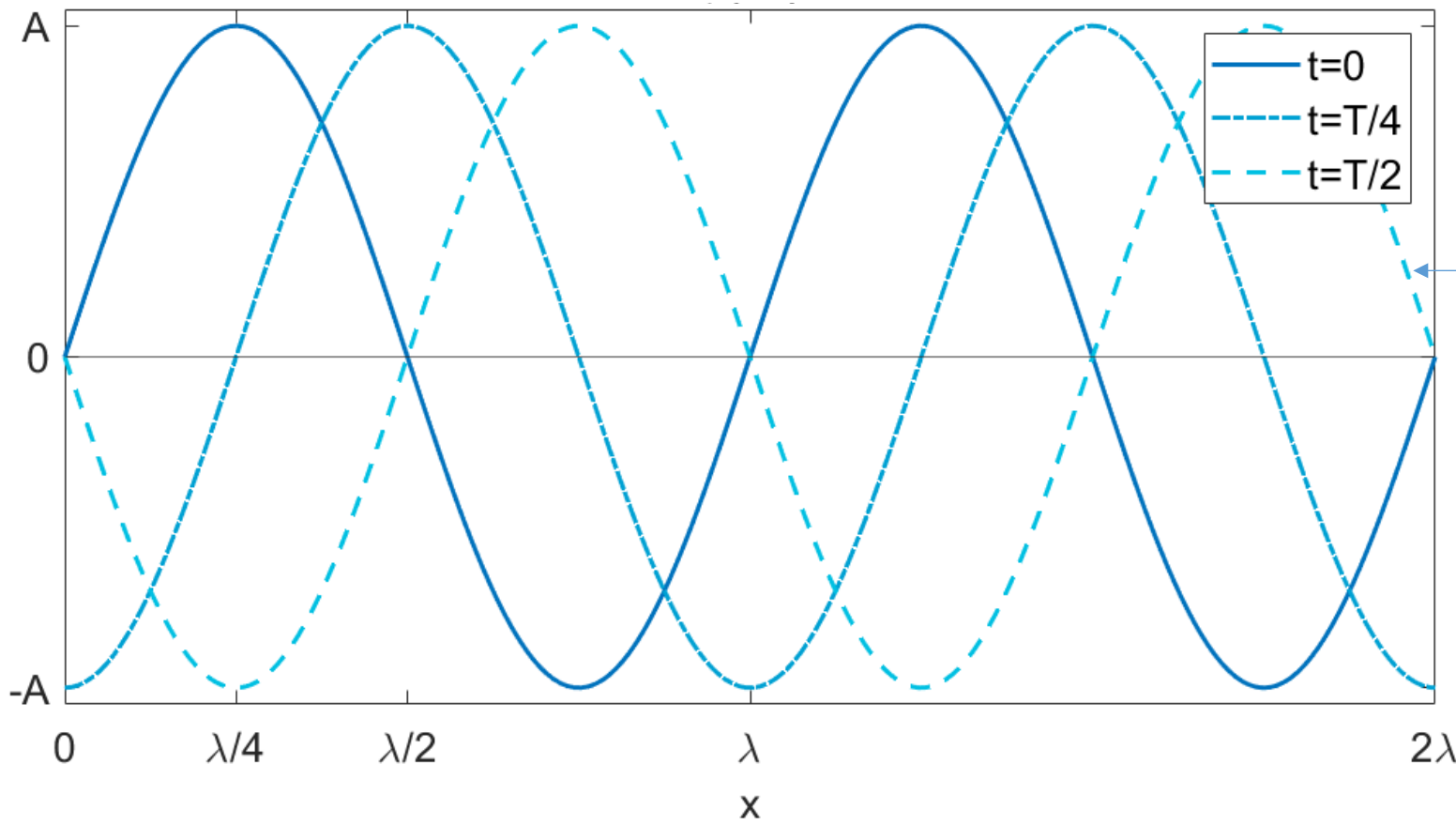
wave vector = radians of oscillation per unit distance

$\omega \equiv 2\pi/T = 2\pi\nu$

angular frequency = radians of oscillation per unit time

Snapshots of our wave at different times

$$\psi(x,t) = A\sin\left(2\pi\left[\frac{x}{\lambda} - \frac{t}{T}\right]\right)$$

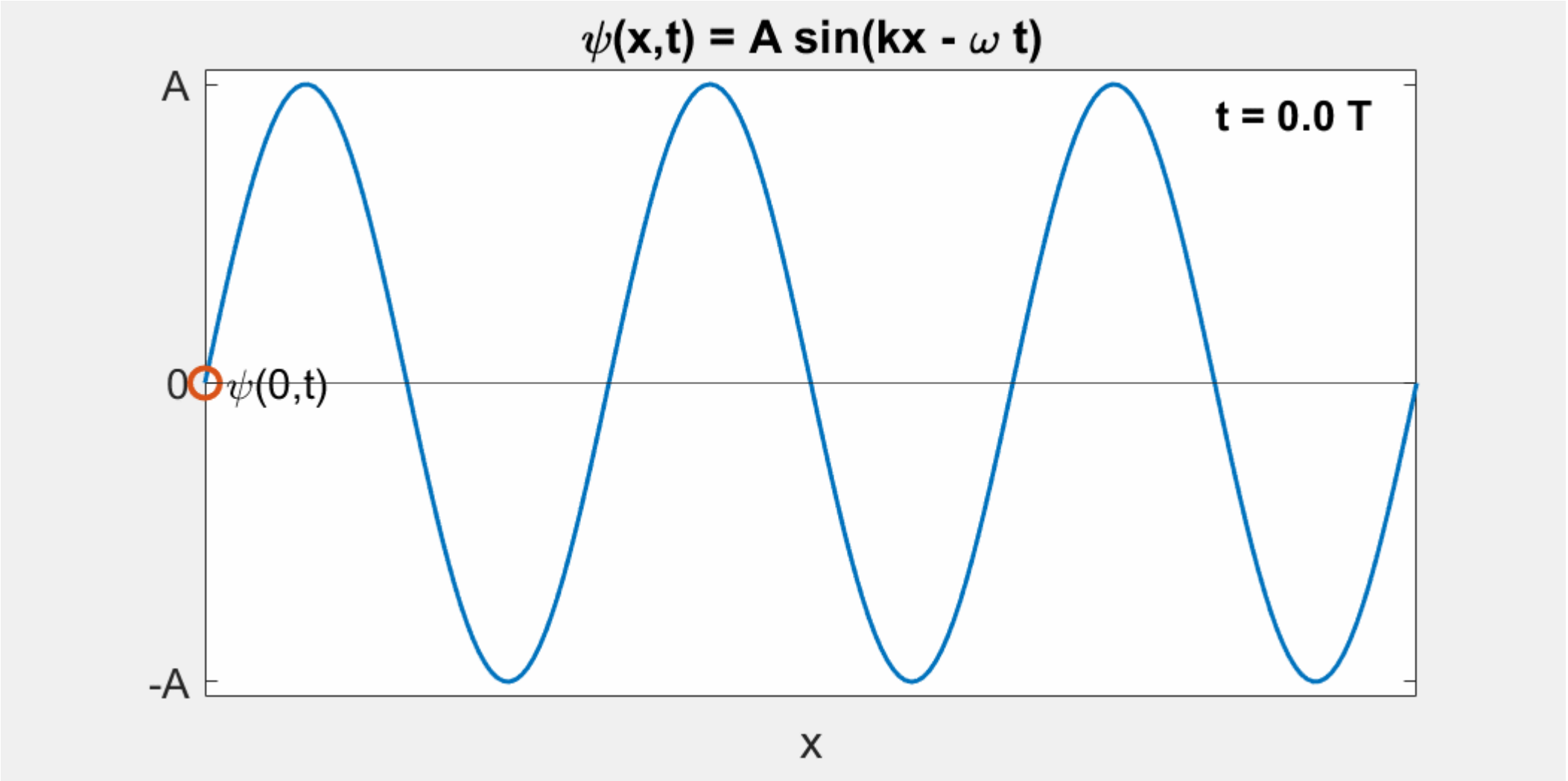


note that the wave appears to propagate to the right in time! this is a ***traveling wave.***

by $t = T/2$, the wave is exactly 180° out of phase with where it started at $t = 0$

after a full period, by $t = T$, it will return to its initial $t = 0$ state

A traveling wave oscillates in both time and space

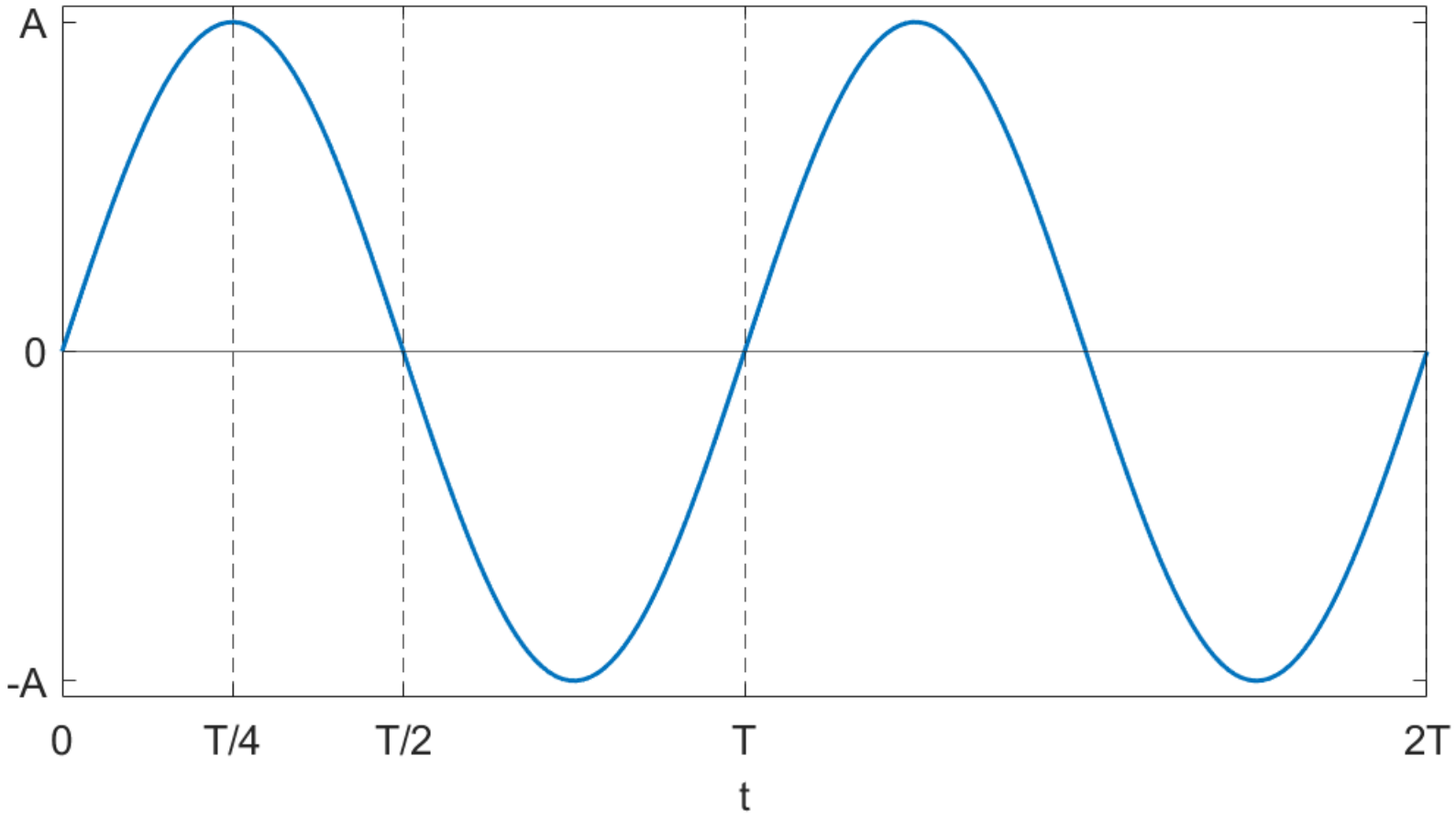


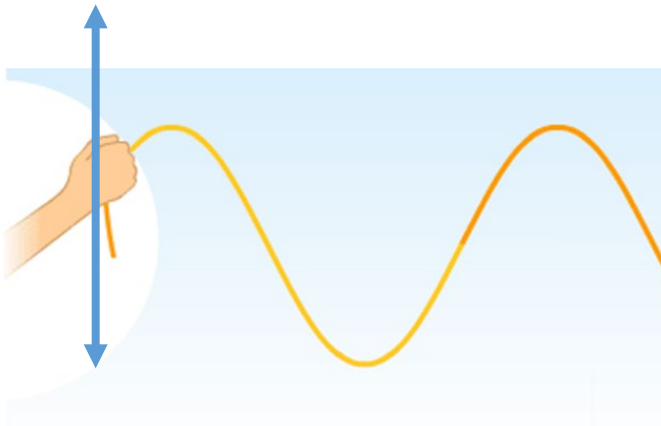
Link to GIF: https://www.dropbox.com/scl/fi/2qs6rz4jej5ql0b3ycbfq/traveling_wave_origin.gif?rlkey=o5qvcxxjx6l2bdjklinyadz3u&dl=0

Oscillations in time at one point in space

$$\psi(x,t) = A\sin\left(2\pi\left[\frac{x}{\lambda} - \frac{t}{T}\right]\right)$$

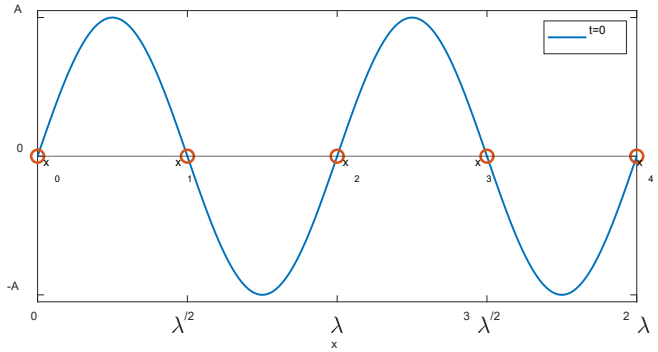
$x = 0$





Finding the velocity of our traveling wave

$$\psi(x,t) = A\sin(kx - \omega t)$$



what is the velocity of this wave? let's see how nodes move in time...

$$\psi(x,t) = A\sin(kx_n - \omega t) = 0$$

nodes are where $\psi(x,t) = 0$,
x_n is the position of the n^{th} node from the origin

$$\therefore kx_n - \omega t = n\pi, \quad n = 0, 1, 2, \ldots$$

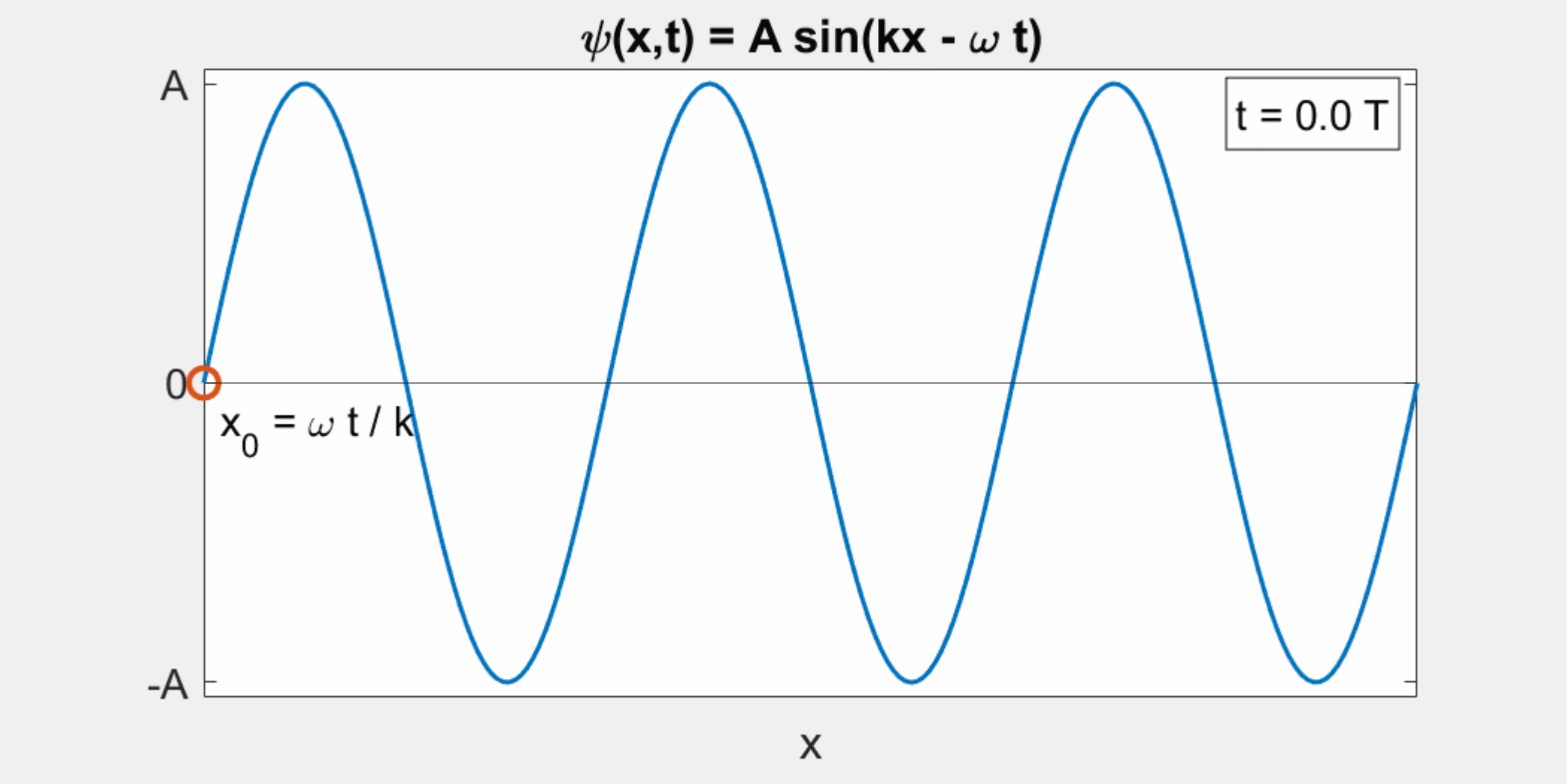
$\sin(y) = 0$ *whenever* $y = n\pi$

$$\rightarrow \; x_n = \frac{n\pi}{k} + \frac{\omega}{k}t \equiv x_n(0) + v \cdot t$$

with $\boxed{v = \frac{\omega}{k}}$ *wave velocity!*

what is the wavefunction for a wave moving in the opposite direction, with negative velocity along the *x*-axis?

Finding the velocity of a traveling wave



Link to GIF: https://www.dropbox.com/scl/fi/z2x0hw5zmcpmo5csc34xt/traveling_wave_velocity.gif?rlkey=kg8y1zl6j8kb6g2khkmb1a8o7&dl=0

An aside: complex exponential notation

Our general expression for a right-going traveling wave: $\psi(x,t) = A\sin(kx - \omega t)$

We will often see wavefunctions written in an equivalent ***complex exponential*** form:

$$\psi(x,t) = Be^{i(kx-\omega t)} + Ce^{-i(kx-\omega t)}$$

We will get comfortable switching back and forth between sines and cosines and complex exponentials using Euler's relations:

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \longleftrightarrow \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

this complex notation will be especially useful in QM where waves can have complex amplitudes

The classical wave equation

The classical wave equation in one dimension

we've now examined what wavefunctions might look like for a classical wave traveling down a length of rope

these wavefunctions are solutions to a differential equation that relates their spatial and temporal structure

note: we make the assumption this equation is "non-dispersive," meaning that wave speed v is a constant for all waves traveling through the medium, e.g. independent of their wavelength and frequency

the classical non-dispersive wave equation

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{1}{v^2} \cdot \frac{\partial^2 \psi(x,t)}{\partial t^2}$$

$v \sim$ speed with which disturbances travel down the rope (depends on material properties e.g. rope mass, tension)

next step: let's double check that the traveling waves we've looked at so far are valid solutions to this differential equation

$$\psi(x,t) = A\sin(kx - \omega t) \quad \textbf{?}$$

Testing our solution to the classical wave equation

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{1}{v^2} \cdot \frac{\partial^2 \psi(x,t)}{\partial t^2} \qquad \psi(x,t) = A \sin(kx - \omega t)$$

recall, derivatives of sine ...

with respect to x

$$\frac{d^2}{dx^2} \sin(kx - \omega t) = \frac{d}{dx} [k \cos(kx - \omega t)] = -k^2 \sin(kx - \omega t)$$

with respect to t

$$\frac{d^2}{dt^2} \sin(kx - \omega t) = \frac{d}{dt} [-\omega \cos(kx - \omega t)] = -\omega^2 \sin(kx - \omega t)$$

Testing our solution to the classical wave equation

$$\frac{\partial^2\psi(x,t)}{\partial x^2} = \frac{1}{v^2}\cdot\frac{\partial^2\psi(x,t)}{\partial t^2} \qquad \psi(x,t) = A\sin(kx-\omega t)$$

left-hand side

$$\frac{\partial^2\psi(x,t)}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left[A\sin(kx-\omega t)\right]$$

$$= -k^2 A\sin(kx-\omega t)$$

$$= \boxed{-k^2\psi(x,t)}$$

right-hand side

$$\frac{1}{v^2}\frac{\partial^2\psi(x,t)}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\left[A\sin(kx-\omega t)\right]$$

$$= \frac{1}{v^2}\left[-\omega^2 A\sin(kx-\omega t)\right]$$

$$= -\frac{\omega^2}{v^2}\psi(x,t)$$

$$= \boxed{-k^2\psi(x,t)}$$

recall our earlier expression for wave speed

$$v = \frac{\omega}{k}$$

so indeed, this wavefunction for a traveling wave is a valid solution to wave equation!

One quick aside on the linearity of the wave equation

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{1}{v^2} \cdot \frac{\partial^2 \psi(x,t)}{\partial t^2}$$

- this is a **linear** partial differential equation because $\psi(x,t)$ and its derivatives only appear to their first power
- a very important property of linear differential equations is that any linear combination of solutions is also a valid solution

for instance, if

$$\psi_1(x,t),\ \psi_2(x,t)$$

are both solutions to a linear differential equation, then any arbitrary linear combination will also satisfy the wave equation:

$$A\psi_1(x,t) + B\psi_2(x,t)$$

Practice problem: a simple differential equation

Consider the differential equation: $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 0$

We want to find a solution $y(x)$

Let's make the guess that: $y(x) = e^{\alpha x}$

Solve for α and write down a general solution to this differential equation.

Hint: α *may be a complex number!*

Can you sketch or describe conceptually how your solution $y(x)$ behaves?

Practice problem solution

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 0 \qquad y(x) = e^{\alpha x}$$

$$\frac{d^2}{dx^2}\left[e^{\alpha x}\right] + 2\frac{d}{dx}\left[e^{\alpha x}\right] + 10\left[e^{\alpha x}\right] = 0$$

$$\alpha^2 e^{\alpha x} + 2\alpha e^{\alpha x} + 10 e^{\alpha x} = 0$$

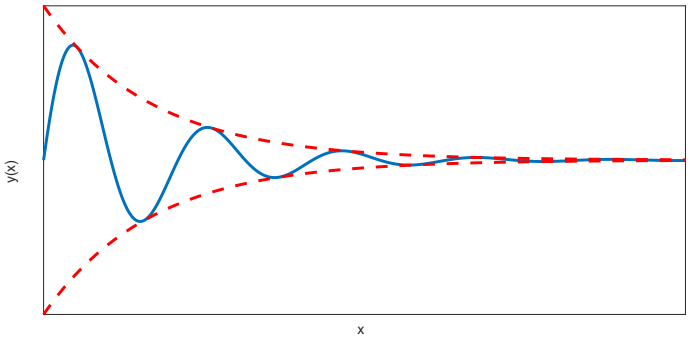
$$\alpha^2 + 2\alpha + 10 = 0$$

use quadratic formula

$$\alpha_{\pm} = \frac{-2 \pm \sqrt{2^2 - 4 \times 10}}{2} = -1 \pm 3i$$

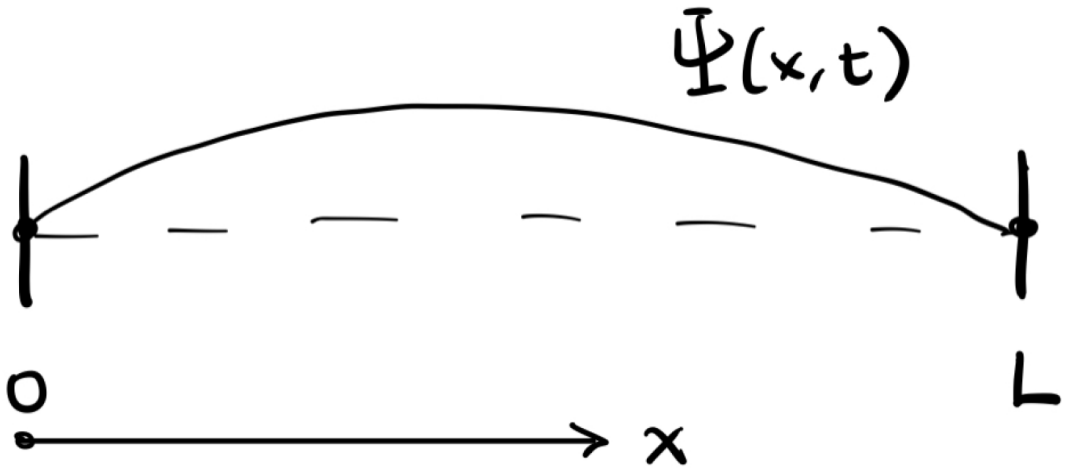
- ***now plug values of α into y(x)***
- ***the most general solution is an arbitrary linear combination of our two solutions***

$$\begin{aligned} y(x) &= Ae^{\alpha_+ x} + Be^{\alpha_- x} \\ &= Ae^{(-1+3i)x} + Be^{(-1-3i)x} \\ &= e^{-x}\left[Ae^{3ix} + B^{-3ix}\right] \\ &= e^{-x}\left[C\cos(3x) + D\sin(3x)\right] \end{aligned}$$



Waves subject to boundary conditions

Waves with **boundary conditions**: the jump rope



boundary conditions

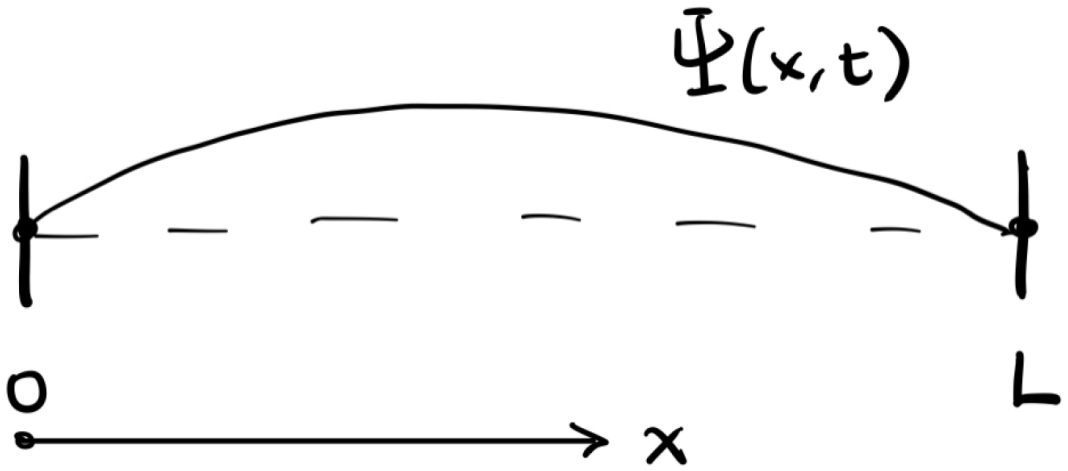
$$\psi(0,t) = 0$$

$$\psi(L,t) = 0$$

for all values of t

let's now consider what waves can propagate in a length of rope of length *L* pinned down at both ends

Waves with **boundary conditions**: the jump rope



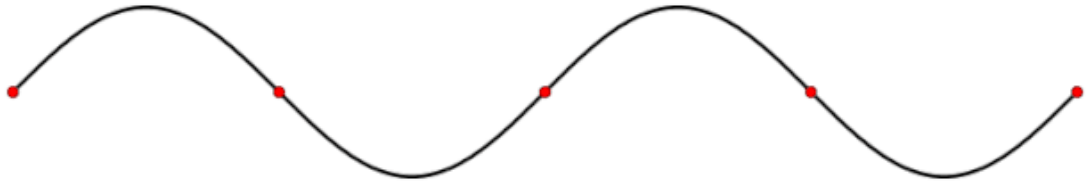
boundary conditions

$$\psi(0,t) = 0$$

$$\psi(L,t) = 0$$

for all values of t

- we will search for ***standing wave*** solutions
- by fixing the ends of the rope, we have made it more natural to think in terms of waves with spatially fixed nodes

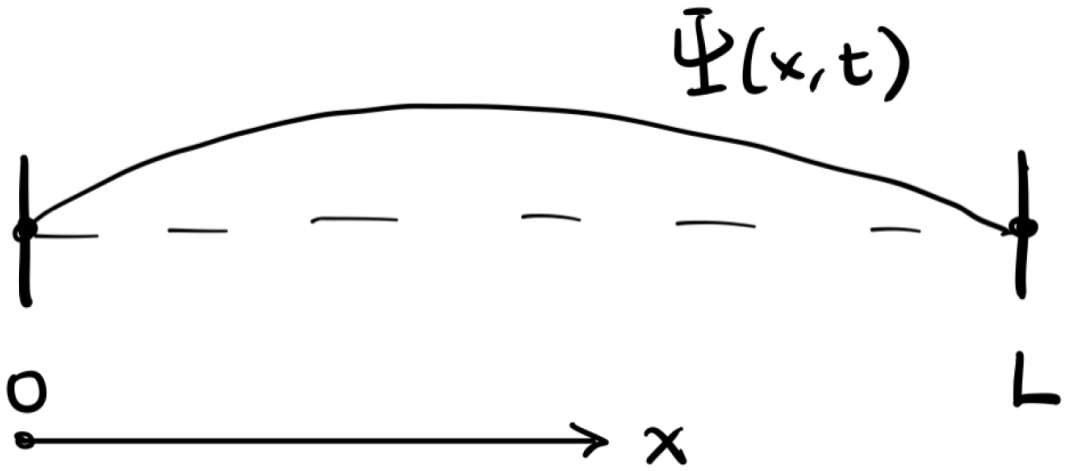


standing waves have ***separable wavefunctions***

$$\psi(x,t) = X(x) \cdot T(t)$$

fixed spatial envelope ***time-dep. modulation***

Waves with **boundary conditions**: the jump rope



note: it may seem like we may only find a subset of possible solutions by searching only for standing waves… but it turns out these will form a ***basis set*** we can use to make linear combinations and construct all possible solutions

- we will search for ***standing wave*** solutions
- by fixing the ends of the rope, we have made it more natural to think in terms of waves with spatially fixed nodes



standing waves have ***separable wavefunctions***

$$\psi(x,t) = X(x) \cdot T(t)$$

fixed spatial envelope *time-dep. modulation*

Solving the wave equation with boundary conditions (B.C.)

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{1}{v^2} \cdot \frac{\partial^2 \psi(x,t)}{\partial t^2}$$

our standing wave guess

$$\psi(x,t) = X(x) \cdot T(t)$$

$$\frac{\partial^2}{\partial x^2} X(x)T(t) = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} X(x)T(t)$$

$$= T(t) \cdot \frac{\partial^2 X(x)}{\partial x^2} = \frac{1}{v^2} X(x) \cdot \frac{\partial^2 T(t)}{\partial t^2}$$

rearrange to collect x terms on LHS, t terms on RHS



$$\rightarrow \frac{1}{X(x)} \cdot \frac{\partial^2 X(x)}{\partial x^2} = \frac{1}{v^2} \frac{1}{T(t)} \cdot \frac{\partial^2 T(t)}{\partial t^2}$$

Wave eqn. with B.C.: separating temporal and spatial variables

$$\frac{1}{X(x)} \cdot \frac{\partial^2 X(x)}{\partial x^2} = \frac{1}{v^2} \frac{1}{T(t)} \cdot \frac{\partial^2 T(t)}{\partial t^2} = K$$

LHS and RHS can only be equal for all values of x, t if they are constant

LHS: purely a function of x

RHS: purely a function of t

separation constant

spatial diff. eq.

$$\frac{1}{X(x)} \cdot \frac{\partial^2 X(x)}{\partial x^2} = K$$

$$\rightarrow \quad \boxed{\frac{\partial^2 X}{\partial x^2} - KX = 0}$$

temporal diff. eq.

$$\frac{1}{v^2} \frac{1}{T(t)} \cdot \frac{\partial^2 T(t)}{\partial t^2} = K$$

$$\rightarrow \quad \boxed{\frac{\partial^2 T}{\partial t^2} - Kv^2 T = 0}$$

Wave eqn. with B.C.: solving the spatial piece

$$\frac{\partial^2 X}{\partial x^2} - KX = 0$$

- we are looking for solutions $X(x)$
- sines and cosines are proportional to their own second derivatives, so let's make an informed guess:

$$X(x) = A\sin(\beta x) + B\cos(\beta x)$$

***goal*: find values for *A*, *B*, *β* that make this guess a valid solution**

$$\frac{d^2}{dx^2}\left[A\sin(\beta x) + B\cos(\beta x)\right] - K\left[A\sin(\beta x) + B\cos(\beta x)\right] = 0$$

$$= -\beta^2\left[A\sin(\beta x) + B\cos(\beta x)\right] - K\left[A\sin(\beta x) + B\cos(\beta x)\right] = 0$$

$$= -(\beta^2 + K)\left[A\sin(\beta x) + B\cos(\beta x)\right] = 0$$

$$\rightarrow \boxed{\beta^2 = -K}$$

what have we learned so far?

- *our guess for $X(x)$ is fine, provided β and K have the correct relationship*
- *if we want β to be real, then we must have $K < 0$*

Wave eqn. with B.C.: applying spatial B.C. to our solution

$$X(x) = A\sin(\beta x) + B\cos(\beta x)$$

apply boundary condition #1: $X(0) = 0$

$$A\sin(0) + B\cos(0) = 0$$

$$A \cdot 0 + B \cdot 1 = 0 \quad \rightarrow \quad \boxed{B = 0}$$

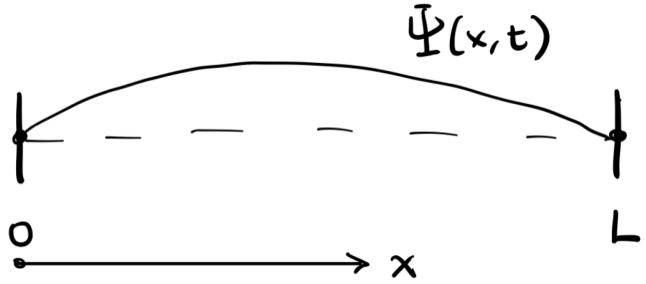
apply boundary condition #2: $X(L) = 0$

$$\rightarrow \quad A\sin(\beta L) = 0$$

$$\beta L = n\pi, \quad n = 1, 2, \ldots \quad \rightarrow \quad \boxed{\beta = \frac{n\pi}{L}}$$

$$\psi(x,t) = X(x) \cdot T(t)$$

recall our boundary conditions:

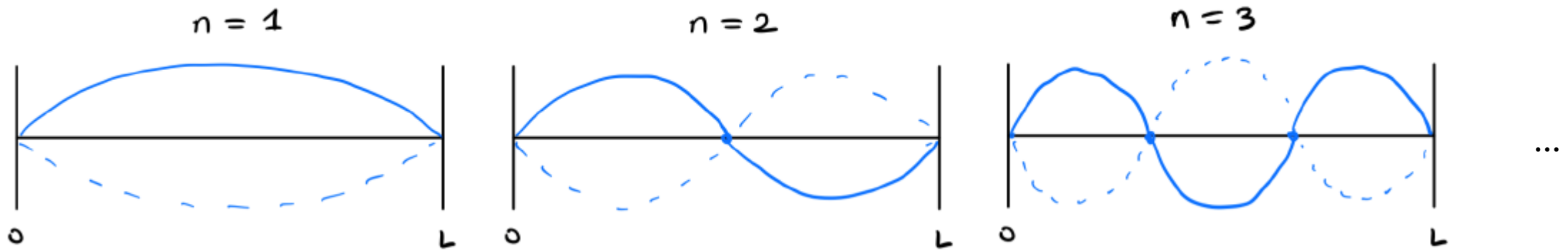


$$\psi(0,t) = 0 \ , \ \psi(L,t) = 0$$

$$X_n(x) = A\sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \ldots$$

Wave eqn. with B.C.: standing wave spatial solutions

$$X_n(x) = A \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \ldots$$



"fundamental" ***"harmonics" or "overtones"*** →

- infinite series of spatial solutions, each with one more half-wavelength squeezed in
- n^{th} solution has $n - 1$ spatial nodes

Wave eqn. with B.C.: solving the temporal piece

$$\frac{\partial^2 T}{\partial t^2} - Kv^2 T = 0 \quad = \quad \frac{d^2 T}{dt^2} + \beta^2 v^2 T \quad = \quad \frac{d^2 T}{dt^2} + \left(\frac{n\pi v}{L}\right)^2 T \quad = 0$$

temporal diff. eq.

$$\boxed{\beta^2 = -K}$$

$$\boxed{\beta = \frac{n\pi}{L}}$$

we now recognize that diff. eqs. of this form always have sinusoidal solutions…

$$T_n(t) = C \sin\left(\frac{n\pi v}{L} t\right) + D \cos\left(\frac{n\pi v}{L} t\right), \quad n = 1, 2, \ldots$$

$$\equiv C \sin(\omega_n t) + D \cos(\omega_n t) \quad \equiv E \sin(\omega_n t + \phi)$$

$$\boxed{\omega_n = \frac{n\pi v}{L}}$$

define

arbitrary phase that captures state of system at $t = 0$

Wave eqn. with B.C.:
putting the spatial and temporal parts together

$$\psi_n(x,t) = X_n(x)T_n(t)$$

$$= A\sin\left(\frac{n\pi}{L}x\right)\cdot E\sin(\omega_n t+\phi)$$

$$= \boxed{F\cdot\sin\left(\frac{n\pi}{L}x\right)\cdot\sin(\omega_n t+\phi)}$$

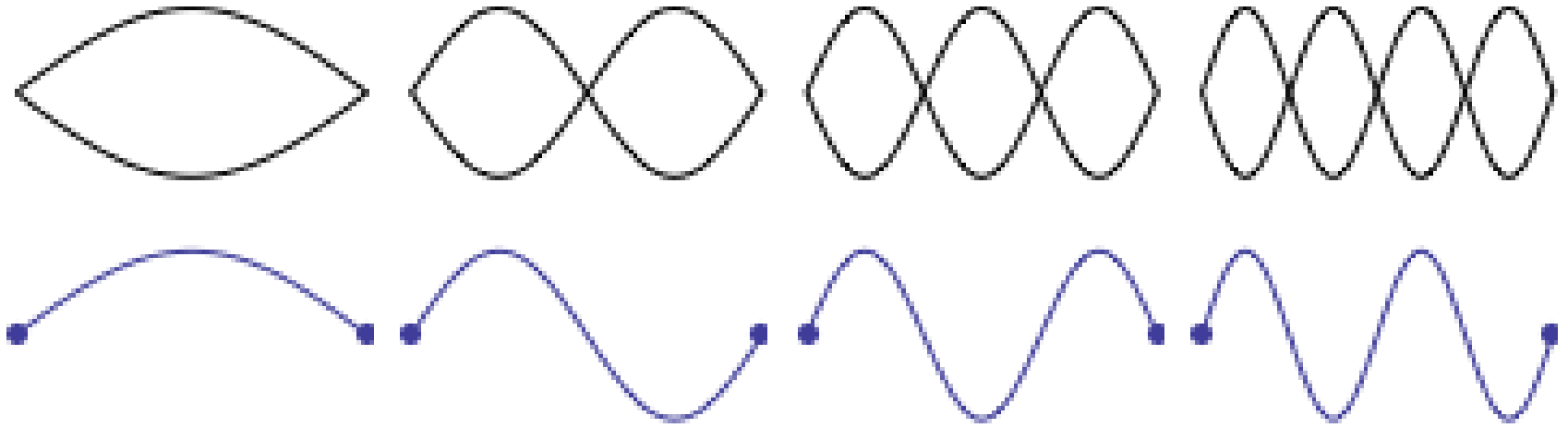
$$\omega_n = \frac{n\pi v}{L}$$

arbitrary scaling factor that is often chosen to "normalize" the wavefunction - more on this in a few lectures

phase that determines state of the wave at time $t = 0$

Standing wave “normal modes”:
faster oscillations in both time and space as *n* increases

$n=1$	$n=2$	$n=3$	$n=4$
“first harmonic”	“second harmonic”	“third harmonic”	“fourth harmonic”



Gif source: https://www.acs.psu.edu/drussell/Demos/Pluck-Fourier/Pluck-Fourier.html

$$X(x) = \sin\left(\frac{\pi}{L}x\right) \quad \sin\left(\frac{2\pi}{L}x\right) \quad \sin\left(\frac{3\pi}{L}x\right) \quad \sin\left(\frac{4\pi}{L}x\right)$$

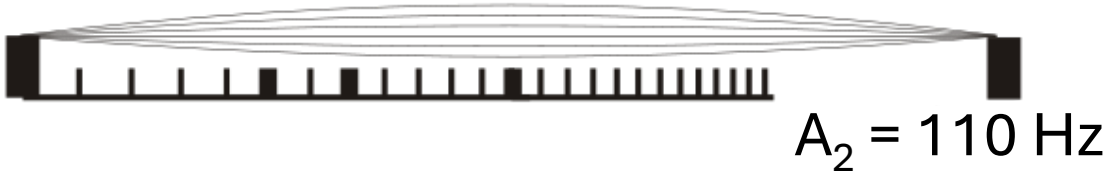
$$T(t) = \sin\left(\omega_1 t + \phi_1\right) \quad \sin\left(\omega_2 t + \phi_2\right) \quad \sin\left(\omega_3 t + \phi_3\right) \quad \sin\left(\omega_4 t + \phi_4\right)$$

$$\omega_1 = \frac{\pi v}{L} \quad \omega_2 = 2\omega_1 \quad \omega_2 = 3\omega_1 \quad \omega_2 = 4\omega_1$$

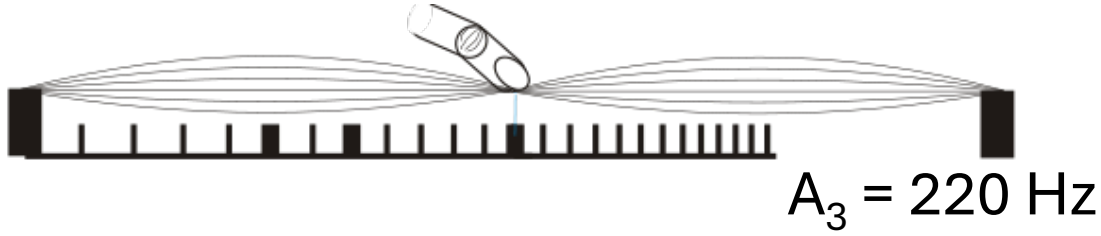
Note #1: the spatial and temporal structure of standing waves is closely related

the standing wave with index n has $n-1$ spatial nodes and oscillates with frequency $\omega_n \propto n$

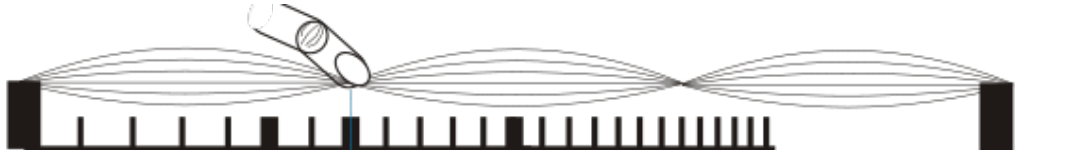




A_2 = 110 Hz



A_3 = 220 Hz



E_4 = 330 Hz



A_4 = 440 Hz

Note #2: standing waves form a basis set with which to construct arbitrary waveforms

- the wave equation is linear, so any linear combination of its solutions is also a valid solution
- we can understand how an arbitrary wave $\Phi(x,t)$ behaves by putting it in terms of our standing wave basis (for $x \subset [0, L]$):

$$\Phi(x,t) = \sum_{n=1}^{\infty} A_n \psi_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) \sin(\omega_n t + \phi)$$

the coefficients in this "basis expansion" determine what the arbitrary wave looks like. more on this later!

***note**: you can extend the musical analogy and think of these linear combination as "chords"*

Superpositions (or “chords”) also satisfy the wave eqn.



$\psi_{16}(x,t)$

$\psi_{20}(x,t)$

$\psi_{24}(x,t)$

you can check: the chord obeys the wave equation and the B.C.... but is not a standing wave!

$$\Phi(x,t) \propto \psi_{16}(x,t) + \psi_{20}(x,t) + \psi_{24}(x,t)$$

Link to GIF: https://www.dropbox.com/scl/fi/1ag9gnirg2tkeh1xc93ng/chords.gif?rlkey=lbi53pvi1qb4jcxnmt9glfvzh&dl=0

Note #3: classical standing waves are the best analogy for solutions to the quantum wave equation!

- quantum objects have wavefunctions which can be found in special states called “eigenstates” which behave just like standing waves: they have separable, closely related temporal and spatial oscillations

- standing waves are a nice basis set to describe how more complex “chords” evolve in space and time



quantum eigenstates will form a nice basis set to describe how complex quantum “superpositions” evolve in space and time

Wrapping up

- **Next time:** Schrödinger and the quantum wave equation.
  Keep the classical waves we considered today in mind as an analogy!

- **Problem Set 1:** posted on Canvas, due next Monday 9/14 at midnight

- **Come to precepts and office hours!**

- **In the last minute**: what is your biggest takeaway from today's lecture?