Two-level systems still display many interesting quantum phenomena. We will assume that we have a system with just two relevant energy eigenstates that lie relatively close together, while all other states are far away in energy and non-interacting. These notes are drawn largely from Cohen-Tannoudji Vol. 1 Chapter 4.

1 Spin-1/2 systems

The prototypical example of a two-level system is a particle with spin $S = 1/2$, like a silver atom with one unpaired electron, as used in the Stern-Gerlach experiments. The two spin levels are degenerate in free space, but will split in energy in an applied magnetic field.

Let’s quickly recall the Stern-Gerlach experiments, where a beam of silver atoms is deflected by a permanent magnet $\vec{B}$ whose field is oriented along the $\hat{z}$ axis:

The atoms deflect because of torque applied by $\vec{B}$ to the magnetic moment of the electron, $\vec{M}$. The magnetic moment is given by

$$\vec{M} = \gamma \vec{S}$$

where $\vec{S}$ is the spin, and $\gamma$ is the gyromagnetic ratio. The energy of the system is closely related, with

$$\Delta E = -\vec{M} \cdot \vec{B}$$

so a spin aligned with the magnetic field experiences a stabilized energy, and one anti-aligned is destabilized.

Classically, we might have expected a continuous spectrum of deflections in this experiment, but the quantum result is that you see two spots with discrete deflections, coming from the quantized two-level spin of the electron. As chemists, we know well that electrons come in two “spin-up” and “spin-down” flavors.
1.1 Matrix Representation of Spin-1/2 Systems

In the $\vec{B} = B \cdot \hat{z}$ field, we are measuring the projection of the electron spin on the z-axis, $\hat{S}_z$.

Let’s think about the eigenvalues and eigenvectors of spin operators. We know the electron has total spin quantum number $S = 1/2$, and that spin comes in units of $\hbar$. Working in the basis of $\hat{S}_z$ eigenstates:

$$\hat{S}_z |+z\rangle = \frac{\hbar}{2} |+z\rangle$$
$$\hat{S}_z |-z\rangle = -\frac{\hbar}{2} |-z\rangle$$

And therefore we can write:

$$\hat{S}_z = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad |+z\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |-z\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

What about the other Cartesian components of the spin $\hat{S}$? $\hat{S}$ behaves like an angular momentum and its components $[\hat{S}_x, \hat{S}_y, \hat{S}_z]$ therefore do not commute. All Cartesian components of the spin share the same eigenvalues, but will have distinct eigenvectors. The $2 \times 2$ matrix representations of these operators are typically represented as the Pauli spin matrices, all expressed within the basis of eigenvectors of $\hat{S}_z$:

$$\hat{S}_z = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \hat{S}_x = \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \hat{S}_y = \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

The total spin can be constructed from these Cartesian components:

$$\hat{S}^2 = \hat{S} \cdot \hat{S} = [\hat{S}_x, \hat{S}_y, \hat{S}_z] \cdot [\hat{S}_x, \hat{S}_y, \hat{S}_z] = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$$

$$\begin{align*}
&= 3 \left( \frac{\hbar}{2} \right)^2 \hat{I} = \frac{3}{4} \hbar^2 \hat{I} = \frac{1}{2} \left[ \frac{1}{2} + 1 \right] \hbar^2 \hat{I} \\
&= S(S+1)\hbar^2 \hat{I}, \quad S = \frac{1}{2}
\end{align*}$$

Aside: The Pauli spin matrices crop up in lots of places because they are a useful basis for representing any arbitrary $2 \times 2$ matrix. We can express a $2 \times 2$ matrix with just four complex parameters:

$$\hat{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Equivalently, we can construct $\hat{A}$ as:

$$\hat{A} = \alpha \hat{I} + \beta \hat{S}_z + \gamma \hat{S}_x + \delta \hat{S}_y$$
1.2 Quantum Measurement of Spin-1/2 Systems

Let’s now think about state preparation and quantum measurement in this system. First just thinking about the beam split into two components by the \( B_z \) field, it’s very easy to do quantum state preparation here. We just pick out one spot of our beam, and we know that we have prepared a system in the \( |+\rangle_z \) state.

But what would happen if we took our upper, separated beam of atoms in the \( |+\rangle_z \) state, and then measured their deflection in a second magnetic field, polarized along the \( x \) axis: \( \vec{B} = B_x \cdot \hat{x} \)?

If we consider what quantum state we’ll measure after separation in the \( B_x \) field, we should only observe eigenstates of \( \hat{S}_x \). What do the eigenstates of \( \hat{S}_x \) look like in our \( \hat{S}_z \) basis?

\[
\hat{S}_x |\pm_x\rangle = \lambda_{\pm} |\pm_x\rangle, \quad \hat{S}_x = \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \lambda_{\pm} = \pm \frac{\hbar}{2}
\]  

\[
\lambda_+ : \quad \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{\hbar}{2} \begin{bmatrix} a \\ b \end{bmatrix} \quad \Rightarrow \quad a = b, \quad |+x\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

\[
\lambda_- : \quad \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = -\frac{\hbar}{2} \begin{bmatrix} c \\ d \end{bmatrix} \quad \Rightarrow \quad c = -d, \quad |x\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

Recall that we prepared our system in the \( |+\rangle_z \) state using a \( B_z \) field, and then separated our atoms again in a \( B_x \) field. This is akin to making sequential measurements of our system with the non-commuting \( \hat{S}_z \) and \( \hat{S}_x \) operators. The likelihood that we a particular atom ends up in the \( |+\rangle_x \) state at the end of these series of operations comes from the contribution of the \( |+\rangle_z \) state to \( |+\rangle_z \):

\[
|\langle +_x | +_z \rangle|^2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{2}
\]

Or, put another way

\[
|+_x \rangle = \frac{1}{\sqrt{2}} |+_z \rangle + \frac{1}{\sqrt{2}} |-_z \rangle, \quad |+_x \rangle^2 = \frac{1}{2}
\]

As \( \hat{S}_z \) and \( \hat{S}_x \) do not commute, they do not share eigenvectors. The pristine \( |+\rangle_z \) quantum state that we prepared initially gets scrambled by measurement with \( \hat{S}_x \), and we see there is a 50:50 chance that we read out a \( |+_x \rangle \) or \( |-_x \rangle \) state.

1.3 Larmor Precession

The concept of Larmor precession might be familiar: a magnetic moment or spin aligned along an arbitrary axis \( \vec{n} \) will revolve, or precess, about the axis of an applied magnetic field. Let’s show that this concept arises out of the spin-1/2 framework we’ve built up so far.
Consider a spin aligned along the arbitrary vector $\vec{n}$:

$$\vec{n} = [\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta]$$  \hspace{1cm} (17)

where $\theta$ is the angle $\vec{n}$ makes with the $z$ axis, and $\phi$ is the phase of $\vec{n}$ in the $x, y$ plane.

We can write down the $2 \times 2$ matrix form for the $\hat{S}_n$ spin component, working in our basis of $\hat{S}_z$ eigenstates:

$$\hat{S}_n = \hat{S} \cdot \vec{n} = [\hat{S}_x, \hat{S}_y, \hat{S}_z] \cdot [\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta]$$  \hspace{1cm} (18)

$$ = \sin \theta \cos \phi \cdot \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \sin \theta \sin \phi \cdot \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \cos \theta \cdot \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$  \hspace{1cm} (19)

$$ = \frac{\hbar}{2} \begin{bmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{bmatrix}$$  \hspace{1cm} (20)

One can show that the eigenvalues of $\hat{S}_n$ will again be $\pm \frac{\hbar}{2}$ and that its normalized eigenstates (expressed in the basis of $\hat{S}_z$ eigenstates) are:

$$|+n\rangle = \cos(\theta/2)e^{-i\phi/2} |+z\rangle$$  \hspace{1cm} (21)

$$+ \sin(\theta/2)e^{i\phi/2} |-z\rangle$$

$$|-n\rangle = -\sin(\theta/2)e^{-i\phi/2} |+z\rangle$$  \hspace{1cm} (22)

$$+ \cos(\theta/2)e^{i\phi/2} |-z\rangle$$

Let’s say we prepare a system with spin aligned along $\vec{n}$ in the $|+_n\rangle$ state, in the presence of a $B_z$ field. How will our system evolve in time? To find the time-dependence of the wavefunction, we always want to start by our system in the energy eigenfunctions of the Hamiltonian.

The Hamiltonian here looks like

$$\hat{H} = -\gamma \hat{S} \cdot \vec{B}$$  \hspace{1cm} (23)

$$= -\gamma B_z \hat{S}_z \text{ since } \vec{B} = B_z$$  \hspace{1cm} (24)

$$\rightarrow \hat{H} |\pm_z\rangle = \mp \gamma \frac{\hbar}{2} B_z |\pm_z\rangle \equiv E_{\pm} |\pm_z\rangle$$  \hspace{1cm} (25)

The eigenstates of $\hat{S}_z$ are therefore also the energy eigenstates and will represent the stationary states of the system.

We now can write down our initial wavefunction in terms of these stationary states:

$$|\Psi(t = 0)\rangle = |+_n\rangle = \cos(\theta/2)e^{-i\phi/2} |+_z\rangle + \sin(\theta/2)e^{i\phi/2} |-z\rangle$$  \hspace{1cm} (26)

$$= (\cos(\theta/2)e^{-i\phi/2} + \sin(\theta/2)e^{i\phi/2}) |+_z\rangle$$  \hspace{1cm} (27)
And therefore the temporal evolution of our wavefunction will look like
\[ |\Psi(t)\rangle = \cos(\theta/2) e^{-i\phi/2} |+z\rangle e^{-iE_+ t/\hbar} \]
\[ + \sin(\theta/2) e^{i\phi/2} |-z\rangle e^{-iE_- t/\hbar} \]
\[ = \cos(\theta/2) e^{-i(\phi+2E_+ t/\hbar)/2} |+z\rangle \]
\[ + \sin(\theta/2) e^{i(\phi+2E_- t/\hbar)/2} |-z\rangle \]
\[ = \cos(\theta/2) e^{-i(\phi-\gamma B_z t)/2} |+z\rangle \]
\[ + \sin(\theta/2) e^{i(\phi-\gamma B_z t)/2} |-z\rangle \]

Inspecting this wavefunction, we can pull out how the orientation of its spin moves in time:
\[ \theta(t) = \theta(t = 0) \]
\[ \phi(t) = \phi(t = 0) - \gamma B_z t \]

Recall that the angle $\phi$ represents the longitude, or phase of the spin vector about the $z$ axis, and it is now a linear function of $t$. Therefore, our spin will precess around the $z$ axis with an angular frequency of $\gamma B_z$. The angle of our spin with respect to the $z$ axis, e.g. its latitude, remains unchanged in this simple example. This is Larmor precession.

2 Perturbations of Two-Level Systems

Two-level systems make a very nice platform in which to explore how small perturbations to a system’s Hamiltonian impact its eigenvalues, eigenvectors, and time-dependent behavior.

Consider the Hamiltonian of a 2-level system subject to a small perturbation:
\[ \hat{H} = \hat{H}_0 + \hat{W} = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} + \begin{bmatrix} 0 & W^* \\ W & 0 \end{bmatrix} = \begin{bmatrix} E_1 & W^* \\ W & E_2 \end{bmatrix} \]

Here $\hat{H}_0$ is a reference Hamiltonian whose eigenstates are $|1\rangle, |2\rangle$; the above is expressed in this basis.

The addition of the perturbing off-diagonal terms will change both the eigenvalues and eigenvectors of the system; $E_1, E_2$ will not be the eigenvalues of $\hat{H}$, and $|1\rangle, |2\rangle$ will not be its eigenvectors. This is a nice problem to explore because we can solve it exactly, but the concepts we learn will be adaptable to the study of much more complex systems.

2.1 Eigenvalues and eigenvectors of a perturbed two-level system

Let’s first think about the eigenvalues of the perturbed system:
\[ \det \left( \hat{H} - E_{\pm} \hat{1} \right) = 0 \quad \rightarrow \quad E_{\pm} = \frac{1}{2} [E_1 + E_2] \pm \sqrt{\frac{1}{4} [E_1 - E_2]^2 + |W|^2} \]
\[ \equiv \hat{E} \pm \sqrt{\Delta^2 + |W|^2} \]
\[ \hat{E} \equiv [E_1 + E_2]; \quad \Delta \equiv \frac{1}{2} |E_1 - E_2| \]

It’s helpful to plot these energy eigenvalues as a function of the $\Delta$ parameter:
We that at $\Delta = 0$, the new energy eigenvalues show an avoided crossing for non-zero coupling ($W \neq 0$). That is to say, when $E_1$ and $E_2$ are degenerate in the unperturbed system, the energy levels of the perturbed system will be split by $2W$.

Let’s consider two interesting cases here:

(i) $\Delta \gg |W|$: Let’s consider what happens when the magnitude of the coupling is small relative to the original difference in energies. Consider the splitting between $E_+$ and $E_-$:

$$E_+ - E_- = 2\sqrt{\Delta^2 + |W|^2} = 2\Delta \sqrt{1 + \frac{|W|^2}{\Delta^2}} = 2\Delta \left[1 + \frac{1}{2} \frac{|W|^2}{\Delta^2} + \ldots\right]$$

(40)

where we have used the Taylor expansion

$$\sqrt{1 + x^2} = 1 + \frac{1}{2} x^2 + \ldots$$

(41)

and dropped higher order terms because we have assumed $\frac{|W|^2}{\Delta^2} \ll 1$. In this “small coupling” regime, we can see that the energy levels change with order $|W|$. This is a perturbative regime, where the eigenvectors and eigenvalues of $\hat{H}$ and $\hat{H}_0$ will be similar. We’ll be able to capture this picture with perturbation theory in a few lectures.

(ii) $\Delta \ll |W|$: Now let’s consider a large perturbation:

$$E_+ - E_- = 2\sqrt{\Delta^2 + |W|^2} = 2|W| \sqrt{\frac{\Delta^2}{|W|^2} + 1} = 2|W| \left[1 + \frac{1}{2} \frac{\Delta^2}{|W|^2} + \ldots\right]$$

(42)

Here are our new states split linearly with $|W|$, which we have taken to be much larger than the original splitting $\Delta$. In this case, we will have drastically different eigenenergies and eigenfunctions in the perturbed system than we did in the original system, which we will not be able to capture accurately with perturbation theory.

We have not yet discussed the eigenvectors of this new perturbed system. The new eigenvectors, with

$$\hat{H} |\pm\rangle = E_\pm |\pm\rangle$$

(43)
can be expressed in the original $|1\rangle, |2\rangle$ basis as:

$$|+\rangle = \cos(\theta/2)e^{-i\phi/2} |1\rangle + \sin(\theta/2)e^{i\phi/2} |2\rangle$$

$$|−\rangle = \sin(\theta/2)e^{-i\phi/2} |1\rangle - \cos(\theta/2)e^{i\phi/2} |2\rangle$$

$$\tan \theta \equiv \frac{|W|}{\Delta}; \quad W = |W| \cdot e^{i\phi} \quad (46)$$

**Aside:** Note that the form of these perturbed eigenvectors is identical to the eigenvectors of the $\hat{S}_n$ spin operator that we described in the previous section, albeit with different definitions of $\theta$ and $\phi$. This is not an accident - both $\hat{H}$ in this example and $\hat{S}_n$ in the previous example are essentially arbitrary Hermitian matrices. The most convenient way to write down their normalized eigenvector solutions is the form given in Eqns. 44 and 45, provided you find the correct parametrization of $\theta$ and $\phi$. See Cohen-Tannoudji Complement B IV for a derivation of these expressions.

Let’s quickly look at what happens to the eigenvectors of our perturbed system in the case of a small perturbation ($\Delta \gg |W|$):

$$\frac{|W|}{\Delta} = \tan \theta \approx \theta \ll 1 \quad (47)$$

where we’ve used the Taylor series for the tangent $\theta$: $\tan(x) = x + \ldots$. Therefore $\theta \rightarrow 0$ and

$$\cos(\theta/2) = 1 + (\theta/2)^2 + \ldots$$

$$\sin(\theta/2) = \theta/2 + \ldots \quad (49)$$

So we have

$$|+\rangle \approx e^{-i\phi/2} |1\rangle + \frac{|W|}{2\Delta} e^{i\phi/2} |2\rangle \quad (50)$$

where the eigenvectors are largely unperturbed, and $|+\rangle \approx |1\rangle$ except for the addition of a global phase factor.

And lastly, let’s look at what happens to the eigenvectors of our system in the case of a large perturbation ($\Delta \ll |W|$):

$$\frac{|W|}{\Delta} = \tan \theta \gg 1 \quad (51)$$

and therefore we must have $\theta \rightarrow \frac{\pi}{2}$, since $\tan(\frac{\pi}{2})$ explodes. Under these conditions, we have

$$|+\rangle = \cos(\theta/2)e^{-i\phi/2} |1\rangle + \sin(\theta/2)e^{i\phi/2} |2\rangle$$

$$= \cos(\pi/4)e^{-i\phi/2} |1\rangle + \sin(\pi/4)e^{i\phi/2} |2\rangle$$

$$= \frac{1}{\sqrt{2}} e^{-i\phi/2} |1\rangle + \frac{1}{\sqrt{2}} e^{i\phi/2} |2\rangle \quad (54)$$

And therefore the new eigenvectors are maximally scrambled from those of the unperturbed system.
2.2 Time-dependence of a perturbed two-level system

We finally turn to the question of the time dynamics in a perturbed two-level system, where we can derive exactly a lot of rich quantum phenomena that will also appear in much more complex systems. Let’s say that at time \( t = 0 \) we prepare our system in one of the original eigenstates of the unperturbed system \( \hat{H}_0 \):

\[
|\Psi(t = 0)\rangle = |1\rangle
\]

(55)

What will the time-dependence of this wavefunction look like? As always, we have to re-express \( |\Psi\rangle \) in terms of the stationary states of \( \hat{H} \) in order to easily write down its time-dependence.

\[
|\Psi(t)\rangle = c_+ e^{-iE_+ t/\hbar} |+\rangle + c_- e^{-iE_- t/\hbar} |-\rangle
\]

(56)

We can find the coefficients \( c_+ \) and \( c_- \) by evaluating:

\[
c_+ = \langle + | 1 \rangle = \cos(\theta/2) e^{i\phi/2}
\]

(57)

\[
c_- = \langle - | 1 \rangle = \sin(\theta/2) e^{i\phi/2}
\]

(58)

which we’ve evaluated through inspection of Eqns. 44 and 45.

Therefore:

\[
|\Psi(t)\rangle = e^{i\phi/2} \left[ \cos(\theta/2) e^{-iE_+ t/\hbar} |+\rangle - \sin(\theta/2) e^{-iE_- t/\hbar} |-\rangle \right]
\]

(59)

and if we like we can drop the global phase factor \( e^{i\phi/2} \) without loss of generality.

If we now ask, what is the probability that we find the system in state \( |+\rangle \) at some time \( t \), we evaluate:

\[
\text{Prob}[|+\rangle] = \left| \cos(\theta/2) e^{-iE_+ t/\hbar} \right|^2 = \cos^2(\theta/2) \neq f(t)
\]

(60)

this is not time-dependent, and therefore not so interesting.

A more interesting question to ask is what the probability is of finding the system in one of its the energy eigenfunctions of the unperturbed system at time \( t \). To evaluate this, we’ll need to substitute Eqns. 44 and 45 into Eqn. 59 one more time:

\[
|\Psi(t)\rangle = \cos(\theta/2)e^{-iE_+ t/\hbar} |+\rangle - \sin(\theta/2)e^{-iE_- t/\hbar} |-\rangle
\]

(61)

\[
= \cos(\theta/2)e^{-iE_+ t/\hbar} \left[ \cos(\theta/2)e^{-i\phi/2} |1\rangle + \sin(\theta/2)e^{i\phi/2} |2\rangle \right]
\]

(62)

\[
+ \sin(\theta/2)e^{-iE_- t/\hbar} \left[ \sin(\theta/2)e^{-i\phi/2} |1\rangle - \cos(\theta/2)e^{i\phi/2} |2\rangle \right]
\]

(63)

\[
= |1\rangle \left[ \cos^2(\theta/2)e^{-iE_+ t/\hbar} e^{-i\phi/2} + \sin^2(\theta/2)e^{-iE_- t/\hbar} e^{i\phi/2} \right]
\]

(64)

\[
+ |2\rangle \left[ \sin(\theta/2)\cos(\theta/2)e^{-iE_+ t/\hbar} e^{i\phi/2} - \sin(\theta/2)\cos(\theta/2)e^{-iE_- t/\hbar} e^{i\phi/2} \right]
\]

(65)
Now we can ask what the probability is that we observe the system in \( |2\rangle \) at time \( t \), recalling that we started in \( |1\rangle \) at \( t = 0 \).

\[
\text{Prob}[|2\rangle] = |\langle 2|\Psi(t)\rangle|^2
\]

\[= \left| \sin(\theta/2) \cos(\theta/2) \right|^2 \left| e^{i\phi/2} \right|^2 \left| e^{-iE_+ t/\hbar} - e^{-iE_- t/\hbar} \right|^2
\]

\[= \frac{1}{2} \sin^2 \theta \left[ e^{iE_+ t/\hbar} - e^{iE_- t/\hbar} \right] \left[ e^{-iE_+ t/\hbar} - e^{-iE_- t/\hbar} \right]
\]

\[= \frac{1}{4} \sin^2 \theta \left[ 2 - e^{i(E_+-E_-)t/\hbar} - e^{-i(E_+-E_-)t/\hbar} \right]
\]

\[= \frac{1}{4} \sin^2 \theta \cdot 2 - 2 \cos \left( \frac{E_+-E_-}{\hbar} t \right)
\]

\[= \sin^2 \theta \cdot \sin^2 \left( \frac{E_+-E_-}{2\hbar} t \right)
\]

\[= \sin^2 \theta \cdot \sin^2 \left( \frac{\delta t}{\hbar} \right), \quad \delta \equiv (E_+-E_-)/2
\]

Plotting this expression as a function of time:

These oscillations in the probability of observing the system in the original, unperturbed eigenstates are called “Rabi oscillations.”

We can rewrite Eqn. 72 by plugging in our expressions for \( E_\pm \) in terms of \( E_1, E_2, \) and \( W \) from Eqn. 37 to find what is often called “Rabi’s formula.”

\[
\text{Prob}[|2\rangle] = \frac{4|W|^2}{4|W|^2 + (E_1 - E_2)^2} \sin^2 \left[ \sqrt{4|W|^2 + (E_1 - E_2)^2} \frac{t}{2\hbar} \right]
\]

What is going on here? We prepared an initial state \( |1\rangle \) that was not a stationary state of the perturbed Hamiltonian. There is no sloshing of probability the states in the \( |+\rangle, |-\rangle \) basis, which are stationary states of \( \hat{H} \). But looking at the dynamics in the \( |1\rangle, |2\rangle \) basis, we do see sloshing of state weightings in time.

Another way to think about this is that the perturbation induces a flopping between the \( |1\rangle \) and \( |2\rangle \) states in the original basis, and we can think of \( 4\text{Prob}[|2\rangle] \) as the probability that the population of the initially prepared state \( |1\rangle \) has been driven into \( |2\rangle \) at time \( t \).
Aside: For those of you used to thinking about strong coupling of molecules in optical cavities, the words “Rabi oscillation” already have a particular connotation. What is the connection to the topic we’ve been looking at here? The quick answer is that if you start with an excited molecule in a resonant optical cavity, and ask what the likelihood is that you observe the molecule still excited some time later, you’d observe Rabi oscillations in that probability. Why? Because the excited molecule is not an energy eigenstate of the coupled light-matter system. More on this when we discuss the Jaynes-Cummings model in a couple of weeks.