CHM 305 - Lecture 2 - The Classical Wave Equation

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In this lecture we will review and solve the classical wave equation, a partial differential equation that governs the spatial and temporal behavior of waves. This classical wave equation provides a conceptual background for Schrödinger's "quantum wave equation," which governs the wavefunctions of quantum mechanical matter, and which we will solve in subsequent lectures. While staying within the classical regime, today's lecture will introduce some math that will prove quite useful over the next few lectures. This material is covered in Chapter 2 of McQuarrie.

1 The Vibrating String

The *wavefunction* of a classical wave describes how a wave's amplitude evolves in space and time. Imagine taking a piece of rope in your hand and shaking it so waves travel down its length. We might write down the following wavefunction to describe the propagation of these waves:

$$\psi(x,t) = A\sin\left(2\pi\left[\frac{x}{\lambda} - \frac{t}{T}\right]\right) \tag{1}$$

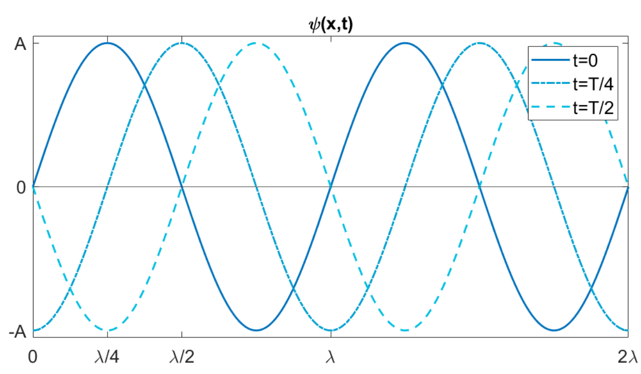
where x is the distance along the direction of wave propagation, λ is the wavelength, and $T = 1/\nu$ is the period.

We will also use an equivalent expression, which is sometimes more convenient, written as:

$$\psi(x,t) = A\sin(kx - \omega t) \tag{2}$$

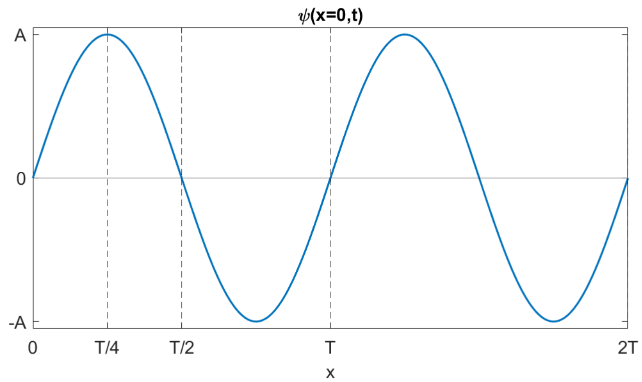
where $k \equiv 2\pi/\lambda$ is the **wave vector** (the number of radians of oscillation the wave undergoes per unit distance, representing a sort of "spatial frequency"), and $\omega \equiv 2\pi/T = 2\pi\nu$ is the **angular frequency** (the number of radians of oscillation the wave undergoes per unit time).

Let's plot various snapshots of this wave to make some sense of it:



These are snapshots of the spatial structure of $\psi(x,t)$ for special values of t. Looking at these snapshots collectively, the wave appears to propagate in time towards the positive end of the x axis. By $t = T/2$ the wave is 180 degrees out of phase from where it started at $t = 0$, and by $t = T$ it will fully return to its initial $t = 0$ state.

We can also consider just one point in space, and think about the sinusoidal wave amplitude at that location over time, as shown below for $x = 0$.



This is a *traveling wave*, meaning that the locations of the peaks and nodes move in time. We can solve for the position of the n^{th} node, x_n (e.g. the n^{th} location from the origin where $\psi(x,t) = 0$) by setting Eqn. 2 to zero, and then examine its time-dependence:

$$\psi(x,t) = A\sin(kx_n - \omega t) = 0 \tag{3}$$

$$\therefore kx_n - \omega t = n\pi \tag{4}$$

$$\rightarrow\ x_n = \frac{n\pi}{k} + \frac{\omega}{k}t \tag{5}$$

$$\equiv x_0 + v \cdot t \tag{6}$$

The nodes are therefore moving with a positive velocity along the x axis of:

$$\boxed{v = \frac{\omega}{k}} \tag{7}$$

It's simple to write down the wavefunction for a wave traveling in the opposite direction, with a negative velocity, simply by switching the sign of either the k or ω term in Eqn. 2:

$$\psi(x,t) = A\sin(kx + \omega t) \tag{8}$$

It's also important to note that we will sometimes use wavefunctions written in an equivalent complex exponential form. Euler's relation tells us that:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{9}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{10}$$

So a general expression for a wave traveling to the right can also be written as:

$$\psi(x,t) = Be^{i(kx-\omega t)} + Ce^{-i(kx-\omega t)} \tag{11}$$

This notation will become especially useful in quantum mechanics where wavefunctions are complex.

2 The Wave Equation in One Dimension

Classical waves are governed by a partial differential equation that relates their spatial structure to their temporal structure – both of which oscillate in time. For waves traveling through a one-dimensional material (like a piece of rope), the relevant wave equation is the **classical non-dispersive wave equation**:

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{1}{v^2} \cdot \frac{\partial^2 \psi(x,t)}{\partial t^2} \tag{12}$$

where v is the speed with which a disturbance or displacement travels down the rope. We make the "non-dispersive" assumption that v is a constant that is the same for all waves, independent of their frequency.

Let's first check that our wave from Eqn. 2 satisfies the wave equation. First, recall that sine and cosine functions are proportional to their own second derivative. In particular:

$$\frac{d^2}{dx^2}\sin(kx-\omega t) = \frac{d}{dx}[k\cos(kx-\omega t)] = -k^2\sin(kx-\omega t) \tag{13}$$

$$\frac{d^2}{dt^2}\sin(kx-\omega t) = \frac{d}{dt}[-\omega\cos(kx-\omega t)] = -\omega^2\sin(kx-\omega t) \tag{14}$$

So plugging our wave from Eqn. 2 into Eqn. 12, we find:

$$\textbf{LHS:} \quad \frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{\partial^2}{\partial x^2}\big[A\sin(kx-\omega t)\big] \tag{15}$$

$$= -k^2 A\sin(kx-\omega t) = \boxed{-k^2\psi(x,t)} \checkmark \tag{16}$$

$$\textbf{RHS:} \quad \frac{1}{v^2}\frac{\partial^2 \psi(x,t)}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\big[A\sin(kx-\omega t)\big] \tag{17}$$

$$= \frac{1}{v^2}\big[-\omega^2 A\sin(kx-\omega t)\big] \tag{18}$$

$$= -\frac{\omega^2}{v^2}\psi(x,t) = \boxed{-k^2\psi(x,t)} \checkmark \tag{19}$$

Where in the last step we used $v = \omega/k$ from Eqn. 7.

Note: Eqn. 12 is a **linear** partial differential equation because $\psi(x,t)$ and its derivatives only appear to the first power. A linear differential equation has the important property that any linear combination of its solutions will also be a solution. For instance, if both $\psi_1(x,t)$ and $\psi_2(x,t)$ satisfy our wave equation, then any linear combination $a\cdot\psi_1(x,t)+b\cdot\psi_2(x,t)$, where a and b are constants, will also satisfy our wave equation. This will become crucial later on!

Practice problem: Consider the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 0$. We will make the guess that $y = e^{\alpha x}$. Solve for α to find the solutions to this system. Can you sketch or describe conceptually how this solution behaves?

3 Waves with Boundary Conditions

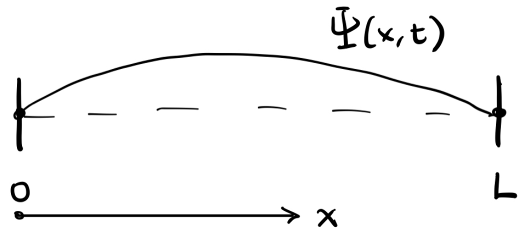
Let's now imagine that instead of just a piece of rope in our hands, we have a length of rope of length L pinned down at both ends – like a jump rope, or a guitar string. You probably have some intuition for what the behavior of this piece of rope will be like, but let's work through a solution to the wave equation to get there.

In particular, we want to solve Eqn. 12 for $\psi(x,t)$ with the conditions that the ends of the rope are pinned down, so:

$$\psi(0,t) = 0 \tag{20}$$

$$\psi(L,t) = 0 \tag{21}$$

for all values of t, as shown below.



We are going to start by making the assumption that $\psi(x,t)$ will be a *standing wave*, not a traveling wave. This is a somewhat intuitive assumption, because by fixing the ends of the rope, we have made it more natural to think in terms of waves with spatially fixed nodes.

Standing waves have *separable* wavefunctions, meaning that the wavefunction factors neatly into two functions of purely x and t:

$$\psi(x,t) = X(x) \cdot T(t) \tag{22}$$

We can therefore think of a standing wave as a fixed spatial envelope given by $X(x)$ with nodes determined by the locations where $X(x) = 0$. The time-dependent part of the wavefunction, $T(t)$, will modulate the amplitude of the envelope so it oscillates in time.

Note: Assuming that our solutions will take on this standing wave form seems like a major constraint, and that we will only recover a small subset of possible solutions! However, it turns out that the standing wave solutions will form a *basis set*, from which we can patch together *any* solution we like. Any possible solution to the wave equation can be described in terms of some linear combination of standing waves – because the wave equation is linear!

So let's get started. Plugging Eqn. 22 into Eqn. 12:

$$\frac{\partial^2}{\partial x^2} X(x)T(t) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2} X(x)T(t) \tag{23}$$

$$= T(t) \cdot \frac{\partial^2 X(x)}{\partial x^2} = \frac{1}{v^2} X(x) \cdot \frac{\partial^2 T(t)}{\partial t^2} \tag{24}$$

$$\rightarrow \frac{1}{X(x)} \cdot \frac{\partial^2 X(x)}{\partial x^2} = \frac{1}{v^2}\frac{1}{T(t)} \cdot \frac{\partial^2 T(t)}{\partial t^2} \tag{25}$$

The left-hand side of Eqn. 25 is now purely a function of x, while the right-hand side is purely a function of t. The only way that equality between the two halves can be preserved for all values of x and t is if both sides are equal to a constant independent of x and t, which we will call the "separation constant" K. It's worth pausing for a moment to make sure this makes sense: both sides of Eqn. 25 must be constant, otherwise, for instance, we could change t to alter the LHS without touching the RHS and ruin the equality. Therefore:

$$\textbf{LHS:} \quad \frac{1}{X(x)} \cdot \frac{\partial^2 X(x)}{\partial x^2} = K \quad \rightarrow \quad \boxed{\frac{\partial^2 X}{\partial x^2} - KX = 0} \tag{26}$$

$$\textbf{RHS:} \quad \frac{1}{v^2}\frac{1}{T(t)} \cdot \frac{\partial^2 T(t)}{\partial t^2} = K \quad \rightarrow \quad \boxed{\frac{\partial^2 T}{\partial t^2} - Kv^2 T = 0} \tag{27}$$

We now have a much simpler problem to solve: two ordinary differential equations that are functions of just one variable. We should already have a good idea for what the solutions will look like. Specifically, we're looking for functions that are proportional to their own second derivative. We have learned that sines and cosines are proportional to their own second derivatives and should do just the trick!

3.1 Finding spatial solutions

Let's take a stab at solving Eqn. 26 for $X(x)$ first. We will make the educated guess that:

$$X(x) = A\sin(\beta x) + B\cos(\beta x) \tag{28}$$

We therefore find:

$$\frac{d^2}{dx^2}\left[A\sin(\beta x) + B\cos(\beta x)\right] - K\left[A\sin(\beta x) + B\cos(\beta x)\right] = 0 \tag{29}$$

$$= -\beta^2\left[A\sin(\beta x) + B\cos(\beta x)\right] - K\left[A\sin(\beta x) + B\cos(\beta x)\right] = 0 \tag{30}$$

$$\rightarrow \boxed{\beta^2 = -K} \tag{31}$$

Since we would like to require that the wave vector (or spatial frequency) β be real, this implies that our separation constant $K < 0$. Otherwise, our guess for $X(x)$ is a perfectly valid solution to our differential equation in Eqn. 26.

Let's consider our boundary conditions now, from Eqns. 20 and 21.

$$X(0) = 0 \quad \rightarrow \quad A\sin(0) + B\cos(0) = 0 \tag{32}$$

$$A \cdot 0 + B \cdot 1 = 0 \tag{33}$$

$$\boxed{B = 0} \tag{34}$$

$$X(L) = 0 \quad \rightarrow \quad A\sin(\beta L) = 0 \tag{35}$$

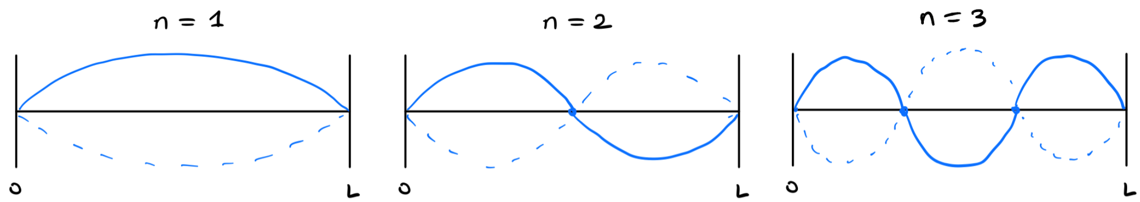
$$\beta L = n\pi, \quad n = 1, 2, \ldots \tag{36}$$

$$\boxed{\beta = \frac{n\pi}{L}} \tag{37}$$

Putting this all together, we now have

$$X_n(x) = A \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \ldots \tag{38}$$

And so we've found the spatial envelopes of our standing waves, which have discrete values allowed for their wave vectors that arose because of the $X(L) = 0$ boundary condition. Here's what the first few look like:



There are an infinite series of these solutions, each designated by index n. We call the $n = 1$, node-less wave the "fundamental," while higher-order waves are "harmonics" or "overtones." These are sinusoids for which a half-integer multiple of the wavelength fits neatly within the box of length 0 to L. Note that each standing wave features $n - 1$ nodes!

3.2 Finding temporal solutions

We can now turn our attention to solving Eqn. 27 for the time-dependent part of the wavefunction given by $T(t)$:

$$\frac{d^2T}{dt^2} - Kv^2T \quad = \quad \frac{d^2T}{dt^2} + \beta^2v^2T \quad = \quad \frac{d^2T}{dt^2} + \left(\frac{n\pi v}{L}\right)^2 T \quad = \quad 0 \tag{39}$$

where we've used the boxed results from Eqns. 31 and 37.

We should now recognize the solutions to differential equations of this form, and we can just write down the answer:

$$T_n(t) = C \sin\left(\frac{n\pi v}{L}t\right) + D \cos\left(\frac{n\pi v}{L}t\right), \quad n = 1, 2, \ldots \tag{40}$$

$$\equiv C \sin(\omega_n t) + D \cos(\omega_n t) \tag{41}$$

$$\equiv E \sin(\omega_n t + \phi) \tag{42}$$

where we have introduced ϕ as an arbitrary phase that captures what the initial state of the system was at $t = 0$, since that was not specified by our boundary conditions.

We have also defined, for simplicity's sake:

$$\omega_n = \frac{n\pi v}{L} \tag{43}$$

3.3 Putting it all together

So finally, our complete time- and position-dependent standing wave solutions are:

$$\psi_n(x,t) = X_n(x)T_n(t) \tag{44}$$

$$= A\sin\left(\frac{n\pi}{L}x\right) \cdot E\sin(\omega_n t + \phi) \tag{45}$$

$$= \boxed{F \cdot \sin\left(\frac{n\pi}{L}x\right) \cdot \sin(\omega_n t + \phi)} \tag{46}$$

where F is an arbitrary constant amplitude that scales the entire wavefunction.

We can now explore a few interesting implications of our results:

(i) **The spatial and temporal structure of standing waves are closely related.** The standing wave with index n has $(n-1)$ spatial nodes and oscillates in place with frequency $\omega_n \propto n$. The denser the spatial oscillations along the length of the rope, the faster the rope must also oscillate in time. This might be familiar if you have played a stringed musical instrument!

(ii) **Standing waves form a basis set to construct arbitrary spatial waveforms.** Because our differential equations are linear, we can take linear combinations, or "superpositions," of our $\psi_n(x,t)$ solutions to construct any arbitrary solution for how the rope might behave:

$$\Phi(x,t) = \sum_{n=1}^{\infty} A_n \psi_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)\sin(\omega_n t + \phi) \tag{47}$$

It turns out that you can construct *any* arbitrary function over the window $x \subset [0, L]$ by choosing the appropriate coefficients A_n. To extend the musical analogy, one can think of these superposition states as "chords."

The use of an infinite collection of sinusoidal functions as a basis set to construct other functions may be familiar if you have seen Fourier analysis in other coursework.

(iii) **Classical standing waves are a very good analogy for solutions to the quantum wave equation.** Keep this idea in mind as we start to think about Schrödinger's equation in the next class.

Quantum objects have states called "eigenstates" which behave exactly like standing waves – they have separable temporal and spatial behavior. Just as standing waves form a basis set to describe how more complex "chords" evolve in time and space, quantum eigenstates serve as a basis set to describe how more complex quantum superposition states evolve.