CHM 502 - Module 4 - Two-Level Systems

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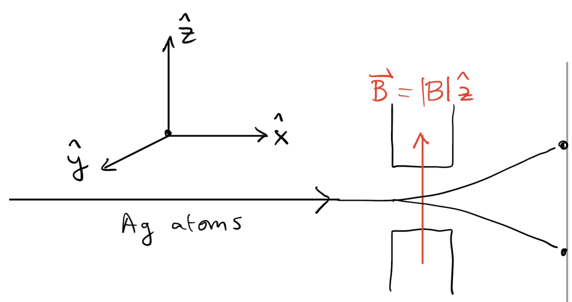
It is amazing how many interesting phenomena can be explored in very simple two-level systems. Here, we will consider a quantum system with just two relevant energy eigenstates lying relatively close together, while all other states are far away in energy and non-interacting.

These notes are drawn in large part from Cohen-Tannoudji Vol. 1, Chapter 4.

1 Spin-1/2 systems

The prototypical example of a two-level system is a particle with spin $S = 1/2$. Consider, for instance, a silver atom with one unpaired electron as used in the famous Stern-Gerlach experiments. A spin-1/2 particle has two spin levels ("spin-up" and "spin-down") which are degenerate in free space but split in energy in an applied magnetic field.

In the Stern-Gerlach experiments a beam of silver atoms is deflected by a permanent magnet whose field is oriented along the $\hat{z}$ axis, with $\vec{B} = |B|\hat{z}$:



A silver atom is deflected from the beam because $\vec{B}$ applies a torque to the atom's magnetic moment $\vec{M}$. The magnetic moment is given by

$$\vec{M} = \gamma\vec{S} \tag{1}$$

where $\vec{S}$ is the spin, and γ is the gyromagnetic ratio. The energy of the system is given by

$$E = -\vec{M}\cdot\vec{B} = -(\gamma\vec{S})\cdot(|B|\hat{z}) = -\gamma|B|\vec{S}\cdot\hat{z} = -\gamma|B|S_z \tag{2}$$

Where S_z is the projection of $\vec{S}$ along the $\hat{z}$ axis. A spin aligned with $\vec{B}$ is stabilized in energy while a spin anti-aligned with the field is destabilized.

Classically, one might expect to observe a continuous spectrum of deflection angles. But the experiment yields two spots with discrete deflections, arising from the quantized two-level electronic spin of the silver atoms.

1.1 Matrix Representation of Spin-1/2 Systems

Let's now consider spin as a quantum operator and examine its eigenvalues and eigenvectors.

Aside: Moving forward we will use terms with arrows (e.g. $\vec{B}$) to represent vectors in 3D space while most terms with hats (e.g. $\hat{S}_z$) represent operators. Things can get muddy because it is common to denote *unit* vectors using hats (e.g. $\hat{z}$, $\hat{n}$). Muddier still, some quantum operators are also vector quantities (e.g. $\hat{S} = [\hat{S}_x, \hat{S}_y, \hat{S}_z]$). Hopefully it is usually clear from context which is which.

Let's start by defining the $\hat{S}_z$ operator corresponding to the S_z observable. One can think of the Stern-Gerlach experiment as applying the $\hat{S}_z$ operator to the wavefunction of our system, and recording which eigenstates of $\hat{S}_z$ are present by the appearance of atoms deflected up and down.

Working in the basis of eigenstates of $\hat{S}_z$ is a good idea since S_z is closely related to the energy of the system (see Eqn. 2). We know that the electron has total spin $S = 1/2$ and that spin comes in units of $\hbar$. We will call the "spin-up" eigenstate $|+_z\rangle$ and the "spin-down" eigenstate $|-_z\rangle$:

$$\hat{S}_z |+_z\rangle = +\frac{\hbar}{2} |+_z\rangle \qquad \hat{S}_z |-_z\rangle = -\frac{\hbar}{2} |-_z\rangle \tag{3}$$

We can therefore construct a nice matrix representation of $\hat{S}_z$:

$$\hat{S}_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad |+_z\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |-_z\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4}$$

What about the other Cartesian components of the total spin operator $\hat{S} = [\hat{S}_x, \hat{S}_y, \hat{S}_z]$? Spin behaves like an angular momentum and its components do not commute. All Cartesian components of the spin share the same eigenvalues $\pm\hbar/2$, but will have distinct eigenvectors. These operators are typically represented as the Pauli spin matrices using the eigenvectors of $\hat{S}_z$ as a basis:

$$\hat{S}_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \hat{S}_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{5}$$

The total spin can be constructed from these Cartesian components:

$$\hat{S}^2 = \hat{S}\cdot\hat{S} = [\hat{S}_x, \hat{S}_y, \hat{S}_z]\cdot[\hat{S}_x, \hat{S}_y, \hat{S}_z] = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 \tag{6}$$

$$= 3\left(\frac{\hbar}{2}\right)^2 \hat{I} = \frac{3}{4}\hbar^2\hat{I} = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2\hat{I} \tag{7}$$

$$= S(S+1)\hbar^2\hat{I}, \qquad S = \frac{1}{2} \tag{8}$$

where we have used the fact that $\hat{S}_x^2 = \hat{S}_y^2 = \hat{S}_z^2 = \left(\frac{\hbar}{2}\right)^2\hat{I}$. You can check that this is the case!

Aside: The Pauli spin matrices crop up in lots of places because they are a useful basis for representing any 2×2 matrix. We can express a 2×2 matrix with just four complex parameters:

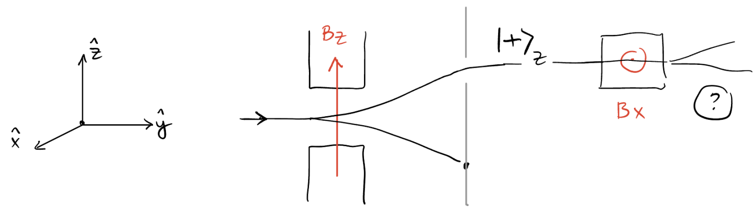
$$\hat{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha\hat{I} + \beta\hat{S}_z + \gamma\hat{S}_x + \delta\hat{S}_y \tag{9}$$

1.2 Quantum Measurement of Spin-1/2 Systems

Let's now think about preparation and measurements of quantum states in two-level systems.

Consider again the beam of atoms split into two components by the B_z field. It's very easy to do quantum state preparation: we just pick out one half of the beam and block the other, and we know that we have prepared a system in, say, the $|+_z\rangle$ state.

Now what would happen if we took our carefully separated beam of atoms in the $|+_z\rangle$ state and ran them through a second magnetic field this time polarized along the x axis, $\vec{B} = |B_x|\hat{x}$?



In making this second separation and measurement along the x axis, we will collapse the system into the eigenvectors of $\hat{S}_x$. Let's examine what the eigenvectors of $\hat{S}_x$ look like in our handy $\hat{S}_z$ basis:

$$\hat{S}_x |\pm_x\rangle = \lambda_\pm |\pm_x\rangle \qquad \hat{S}_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \lambda_\pm = \pm\frac{\hbar}{2} \tag{10}$$

$$\lambda_+ : \quad \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} a \\ b \end{bmatrix} \quad \rightarrow \quad a = b, \quad |+_x\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{11}$$

$$\lambda_- : \quad \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = -\frac{\hbar}{2}\begin{bmatrix} c \\ d \end{bmatrix} \quad \rightarrow \quad c = -d, \quad |-_x\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{12}$$

We initially prepared our system in the $|+\rangle_z$ state then separated our atoms in a B_x field. This is akin to making sequential measurements with the non-commuting $\hat{S}_z$ and $\hat{S}_x$ operators. The likelihood that an atom initially prepared in the $|+\rangle_z$ state ends up in the $|+\rangle_x$ state is given by:

$$\Pr[+_x] = |\langle +_x|+_z\rangle|^2 = \left|\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right|^2 = \frac{1}{2} \tag{13}$$

There is a 50:50 chance that we subsequently read out a $|+_x\rangle$ or $|-_x\rangle$ state. As $\hat{S}_z$ and $\hat{S}_x$ do not commute, they do not share eigenvectors. The pristine $|+\rangle_z$ quantum state that we prepared initially gets scrambled by measurement with $\hat{S}_x$.

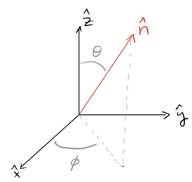
1.3 Larmor Precession

Spins do interesting things in 3 dimensions. Imagine a spin initially aligned along some axis $\vec{n}$ which experiences a magnetic field aligned along some second, distinct axis, say $\vec{B} = |B|\hat{z}$. It will turn out that this spin will revolve, or precess, in time about the axis of the applied magnetic field. We can show how this phenomenon, so-called "Larmour precession" arises from the spin-1/2 framework.

Consider a spin aligned along the unit vector $\hat{n}$:

$$\hat{n} = [\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta] \tag{14}$$

where θ is the polar angle that $\vec{n}$ makes with the z axis, and ϕ is the azimuthal angle of $\hat{n}$ in the x, y plane.



Let write down an operator, which we'll call $\hat{S}_n$, that lets us measure the spin of our system with respect to the $\hat{n}$ axis. We will work in the basis of $\hat{S}_z$ eigenvectors since our magnetic field is aligned along $\hat{z}$:

$$\hat{S}_n = \hat{S}\cdot\hat{n} = [\hat{S}_x, \hat{S}_y, \hat{S}_z]\cdot[\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta] \tag{15}$$

$$= \sin\theta\cos\phi\cdot\frac{\hbar}{2}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + \sin\theta\sin\phi\cdot\frac{\hbar}{2}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix} + \cos\theta\cdot\frac{\hbar}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \tag{16}$$

$$= \frac{\hbar}{2}\begin{bmatrix}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta\end{bmatrix} \tag{17}$$

One can diagonalize $\hat{S}_n$ to find that its eigenvalues are again $\pm\hbar/2$ and its eigenstates (expressed in the basis of $\hat{S}_z$ eigenvectors) are:

$$|+_n\rangle = \cos(\theta/2)\,e^{-i\phi/2}\,|+_z\rangle \quad + \quad \sin(\theta/2)\,e^{i\phi/2}\,|-_z\rangle \tag{18}$$

$$|-_n\rangle = -\sin(\theta/2)\,e^{-i\phi/2}\,|+_z\rangle \quad + \quad \cos(\theta/2)\,e^{i\phi/2}\,|-_z\rangle \tag{19}$$

Let's say we prepare a system with spin aligned along $\hat{n}$ in the $|+_n\rangle$ state, in the presence of a $|B|\hat{z}$ field. How will our system evolve in time? To find the time-dependence of the wavefunction, we always start by laying out our system in the basis of energy eigenfunctions.

The Hamiltonian here looks like

$$\hat{H} = -\gamma\hat{S}\cdot\vec{B} \tag{20}$$

$$= -\gamma\cdot\left[\hat{S}_x, \hat{S}_y, \hat{S}_z\right]\cdot|B|\hat{z} = -\gamma|B|\hat{S}_z \tag{21}$$

$$\rightarrow \hat{H}\,|\pm_z\rangle = \mp\frac{\hbar\gamma|B|}{2}\,|\pm_z\rangle \equiv E_\pm\,|\pm_z\rangle \tag{22}$$

The eigenstates of $\hat{S}_z$ are therefore also the energy eigenstates, which makes it very easy to write down their time depenence.

We now can write down our initial wavefunction in terms of these stationary states:

$$|\Psi(t=0)\rangle \quad = \quad |+_n\rangle \quad = \quad \cos(\theta/2)e^{-i\phi/2}\,|+_z\rangle \quad + \quad \sin(\theta/2)e^{i\phi/2}\,|-_z\rangle \tag{23}$$

And therefore our wavefunction will evolve in time according to:

$$|\Psi(t)\rangle = \cos(\theta/2)e^{-i\phi/2}\,|+_z\rangle\, e^{-iE_+t/\hbar} \quad + \quad \sin(\theta/2)e^{i\phi/2}\,|-_z\rangle\, e^{-iE_-t/\hbar} \tag{24}$$

$$= \cos(\theta/2)e^{-i[\phi+2E_+t/\hbar]/2}\,|+_z\rangle \quad + \quad \sin(\theta/2)e^{+i[\phi+2E_-t/\hbar]/2}\,|-_z\rangle \tag{25}$$

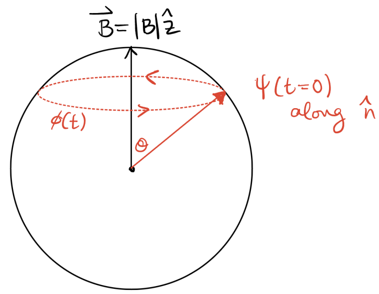
$$= \cos(\theta/2)e^{-i[\phi-\gamma B_z t]/2}\,|+_z\rangle \quad + \quad \sin(\theta/2)e^{+i[\phi-\gamma B_z t]/2}\,|-_z\rangle \tag{26}$$

Inspecting this wavefunction, we can pull out how its spin vector precesses in time:

$$\text{polar angle}: \qquad \theta(t) = \theta(t=0) \tag{27}$$

$$\text{azimuthal angle}: \qquad \phi(t) = \phi(t=0) - \gamma B_z t \tag{28}$$

We find that $\phi(t)$, the longitude or phase of the spin vector about the $\hat{z}$ axis, is a linear function of t. The spin will therefore precess about the $\hat{z}$ axis with an angular frequency of γB_z. θ, the polar angle of the spin with respect to $\hat{z}$ remains fixed in time, since we've included no spin relaxation in this simple example.



2 Perturbations of Two-Level Systems

Two-level systems make a nice platform in which to explore how small perturbations to a Hamiltonian impact the eigenvalues, eigenvectors, and time-dependent behavior of the system.

Consider the Hamiltonian of a 2-level system subject to a small perturbation:

$$\hat{H} = \hat{H}_0 + \hat{W} = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} + \begin{bmatrix} 0 & W^* \\ W & 0 \end{bmatrix} = \begin{bmatrix} E_1 & W^* \\ W & E_2 \end{bmatrix} \tag{29}$$

Here $\hat{H}_0$ is a reference Hamiltonian with eigenstates $\{|1\rangle, |2\rangle\}$ and energy eigenvalues E_1 and E_2. The perturbing off-diagonal terms will change both the eigenvalues and eigenvectors of the system; of $\hat{H}$.

We will solve this problem exactly here, and the concepts we learn will be adaptable to the study of much more complex systems.

2.1 Eigenvalues and eigenvectors of a perturbed two-level system

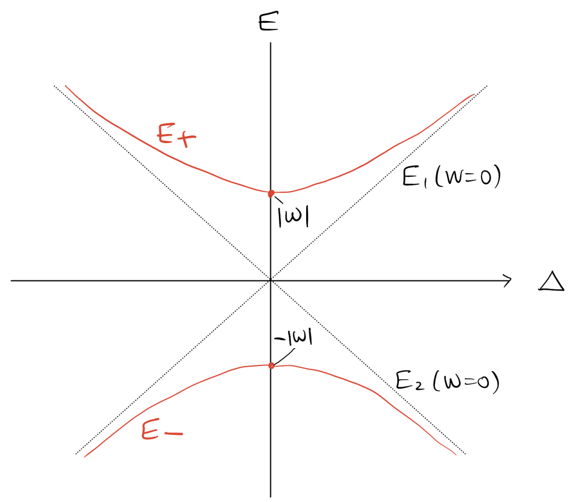
Let's first solve for the new eigenvalues of the perturbed system, which we will call E_+ and E_-:

$$\det\left[\hat{H} - E_\pm \hat{I}\right] = 0 \qquad \rightarrow \qquad E_\pm = \frac{1}{2}\left[E_1 + E_2\right] \pm \sqrt{\frac{1}{4}\left[E_1 - E_2\right]^2 + |W|^2} \tag{30}$$

$$\equiv \bar{E} \pm \sqrt{\Delta^2 + |W|^2} \tag{31}$$

$$\text{where} \qquad \bar{E} \equiv \left[E_1 + E_2\right]; \quad \Delta \equiv \frac{1}{2}\left[E_1 - E_2\right] \tag{32}$$

It's helpful to plot $E_\pm$ as a function of the Δ parameter:



The new energy eigenvalues show an avoided crossing at $\Delta = 0$ for non-zero coupling ($W \neq 0$). More specifically, when the unperturbed energy eigenvalues of E_1 and E_2 are degenerate, the energy levels of the perturbed system will be split by $2W$.

Let's consider two regimes here:

(i) $|\mathbf{W}| \ll \mathbf{\Delta}$: Let's consider what happens when the perturbation is small relative to the difference in energies of the original uncoupled states. Consider the splitting between E_+ and E_-:

$$E_+ - E_- = 2\sqrt{\Delta^2 + |W|^2} = 2\Delta\sqrt{1 + \frac{|W|^2}{\Delta^2}} = 2\Delta\left[1 + \frac{1}{2}\frac{|W|^2}{\Delta^2} + \ldots\right] \tag{33}$$

where we have used the Taylor expansion

$$\sqrt{1 + x^2} = 1 + \frac{1}{2}x^2 + \ldots \tag{34}$$

which is valid for $\frac{|W|^2}{\Delta^2} \ll 1$. In this "weak coupling" regime, the energy levels are static to zeroth order, changing as $|W|^2$. The eigenvectors and eigenvalues of $\hat{H}$ will therefore be quite similar to those of $\hat{H}_0$. One can capture this regime with perturbation theory.

(ii) $|\mathbf{W}| \gg \mathbf{\Delta}$: Now let's consider a large perturbation, taking instead the limit where $\frac{\Delta^2}{|W|^2} \ll 1$:

$$E_+ - E_- = 2\sqrt{\Delta^2 + |W|^2} = 2|W|\sqrt{\frac{\Delta^2}{|W|^2} + 1} = 2|W|\left[1 + \frac{1}{2}\frac{\Delta^2}{|W|^2} + \dots\right] \tag{35}$$

Now the new states split *linearly* with $|W|$, which we have taken to be much larger than the original splitting Δ. In this case, we will have drastically different eigenenergies and eigenfunctions for $\hat{H}$ than we did for $\hat{H}_0$. This is no longer within the "perturbative" regime.

We have not yet discussed the new eigenvectors of this perturbed system, which we will call $|+\rangle$ and $|-\rangle$:

$$\hat{H}\,|\pm\rangle = E_\pm\,|\pm\rangle \tag{36}$$

By evaluating Eqn. 36 by plugging in $\hat{H}$ and $E_\pm$, one can show that these vectors can be expressed in the original $|1\rangle, |2\rangle$ basis as:

$$|+\rangle = \cos(\theta/2)e^{-i\phi/2}\,|1\rangle + \sin(\theta/2)e^{i\phi/2}\,|2\rangle \tag{37}$$

$$|-\rangle = \sin(\theta/2)e^{-i\phi/2}\,|1\rangle - \cos(\theta/2)e^{i\phi/2}\,|2\rangle \tag{38}$$

$$\tan\theta \equiv \frac{|W|}{\Delta} \qquad W = |W| \cdot e^{i\phi} \tag{39}$$

Aside: Note that the form of $|\pm\rangle$ is identical to the eigenvectors of the $\hat{S}_n$ spin operator we described earlier, albeit with different definitions of θ and ϕ. This is not an accident. Both examples are eigenvectors of "arbitrary" 2×2 Hermitian matrices. Eqns. 37 and 38 provide the most convenient way to write down the general, normalized eigenvector solutions to such systems provided you find the correct parametrization of θ and ϕ. See Cohen-Tannoudji Complement $\mathrm{B_{IV}}$ for a derivation of these expressions.

Again, we'll consider two regimes of our perturbed eigenvectors:

(i) $|\mathbf{W}| \ll \mathbf{\Delta}$: Let's look at what happens to $|\pm\rangle$ in the case of a small perturbation:

$$\frac{|W|}{\Delta} = \tan\theta \approx \theta \ll 1 \tag{40}$$

where we've used the Taylor expansion $\tan(x) = x + \dots$ We can therefore take small perturbation limit to be the limit where $\theta \to 0$. This means we can Taylor expand the sine and cosine as well:

$$\cos(\theta/2) = 1 + (\theta/2)^2 + \dots \qquad \sin(\theta/2) = \theta/2 + \dots \tag{41}$$

So we have

$$|+\rangle \approx e^{-i\phi/2}\,|1\rangle + \frac{|W|}{2\Delta}e^{i\phi/2}\,|2\rangle \approx e^{-i\phi/2}\,|1\rangle \tag{42}$$

$$|-\rangle \approx \frac{|W|}{2\Delta}e^{-i\phi/2}\,|1\rangle - e^{i\phi/2}\,|2\rangle \approx -e^{i\phi/2}\,|2\rangle \tag{43}$$

The eigenvectors are largely unperturbed: e.g. $|+\rangle \approx |1\rangle$ except for the addition of a global phase factor.

(ii) $|\mathbf{W}| \gg \mathbf{\Delta}$: Lastly, let's consider the eigenvectors of our system in the case of a large perturbation ($\Delta \ll |W|$):

$$\frac{|W|}{\Delta} = \tan\theta \gg 1 \tag{44}$$

and therefore we must have $\theta \to \frac{\pi}{2}$, since $\tan(\frac{\pi}{2})$ explodes. Under these conditions, we have

$$|+\rangle = \cos(\theta/2)e^{-i\phi/2}\,|1\rangle + \sin(\theta/2)e^{i\phi/2}\,|2\rangle \tag{45}$$

$$= \cos(\pi/4)e^{-i\phi/2}\,|1\rangle + \sin(\pi/4)e^{i\phi/2}\,|2\rangle \tag{46}$$

$$= \frac{1}{\sqrt{2}}e^{-i\phi/2}\,|1\rangle + \frac{1}{\sqrt{2}}e^{i\phi/2}\,|2\rangle \tag{47}$$

And therefore the new eigenvectors are maximally scrambled from those of the unperturbed system.

2.2 Time-dependence of a perturbed two-level system

Let's finally turn to the question of the time dynamics of a perturbed two-level system. Here, we can exactly derive much of the rich quantum phenomena that appear in more complex systems.

Let's say that at time $t = 0$ we prepare our system in one of the original eigenstates of $\hat{H}_0$:

$$|\Psi(t=0)\rangle = |1\rangle \tag{48}$$

To find out how this wavefunction will evolve under the perturbed Hamiltonian $\hat{H}$, we must express $|\Psi\rangle$ in terms of the eigenstates of $\hat{H}$:

$$|\Psi(t)\rangle = c_+ e^{-iE_+t/\hbar}\,|+\rangle + c_- e^{-iE_-t/\hbar}\,|-\rangle \tag{49}$$

We can find c_+ and c_- using our results from Eqns. 37 and 38:

$$c_+ = \langle +|1\rangle = \cos(\theta/2)e^{i\phi/2} \tag{50}$$

$$c_- = \langle -|1\rangle = \sin(\theta/2)e^{i\phi/2} \tag{51}$$

Therefore:

$$|\Psi(t)\rangle = e^{i\phi/2}\left[\cos(\theta/2)e^{-iE_+t/\hbar}\,|+\rangle + \sin(\theta/2)e^{-iE_-t/\hbar}\,|-\rangle\right] \tag{52}$$

We can drop the global phase factor $e^{i\phi/2}$ without loss of generality.

To find the probability that we find the system in state $|+\rangle$ at some time t, we evaluate:

$$\text{Prob}\Big[\,|+\rangle\,\Big] = \Big|\,\langle +|\Psi(t)\rangle\,\Big|^2 = \Big|\cos(\theta/2)e^{-iE_+t/\hbar}\Big|^2 = \cos^2(\theta/2) \neq f(t) \tag{53}$$

This probability is not time-dependent, and therefore not so interesting.

Instead, let's examine the probability of finding the system in one of the *unperturbed* $\hat{H}_0$ Hamiltonian's eigenfunctions at time t. To evaluate this, we substitute Eqns. 37 and 38 into Eqn. 52 once more:

$$|\Psi(t)\rangle = \cos(\theta/2)e^{-iE_+t/\hbar}\,|+\rangle + \sin(\theta/2)e^{-iE_-t/\hbar}\,|-\rangle \tag{54}$$

$$= \cos(\theta/2)e^{-iE_+t/\hbar}\Big[\cos(\theta/2)e^{-i\phi/2}\,|1\rangle + \sin(\theta/2)e^{i\phi/2}\,|2\rangle\Big] \tag{55}$$

$$+ \sin(\theta/2)e^{-iE_-t/\hbar}\Big[\sin(\theta/2)e^{-i\phi/2}\,|1\rangle - \cos(\theta/2)e^{i\phi/2}\,|2\rangle\Big] \tag{56}$$

$$= |1\rangle\left[\cos^2(\theta/2)e^{-iE+t/\hbar}e^{-i\phi/2} + \sin^2(\theta/2)e^{-iE_-t/\hbar}e^{-i\phi/2}\right] \tag{57}$$

$$+ |2\rangle\left[\sin(\theta/2)\cos(\theta/2)e^{-iE_+t/\hbar}e^{i\phi/2} - \sin(\theta/2)\cos(\theta/2)e^{-iE_-t/\hbar}e^{i\phi/2}\right] \tag{58}$$

Now we can find the probability that we find the system in $|2\rangle$ at time t, recalling that we started in $|1\rangle$ at $t = 0$:

$$\text{Prob}\Big[\,|2\rangle\,\Big] = |\langle 2|\Psi(t)\rangle|^2 \tag{59}$$

$$= \big|\sin(\theta/2)\cos(\theta/2)\big|^2\ \cancelto{1}{\big|e^{i\phi/2}\big|^2}\ \big|e^{-iE_+t/\hbar} - e^{-iE_-t/\hbar}\big|^2 \tag{60}$$

$$= \left|\frac{1}{2}\sin\theta\right|^2\ \left[e^{iE_+t/\hbar} - e^{iE_-t/\hbar}\right]\left[e^{-iE_+t/\hbar} - e^{-iE_-t/\hbar}\right] \tag{61}$$

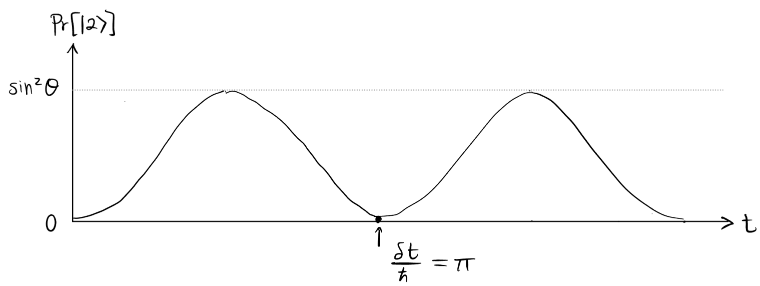
$$= \frac{1}{4}\sin^2\theta\left[2 - e^{i[E_+-E_-]t/\hbar} - e^{-i[E_+-E_-]t/\hbar}\right] \tag{62}$$

$$= \frac{1}{4}\sin^2\theta\left[2 - 2\cos\left(\frac{E_+ - E_-}{\hbar}t\right)\right] \tag{63}$$

$$= \sin^2\theta\cdot\sin^2\left(\frac{E_+ - E_-}{2\hbar}t\right) \tag{64}$$

$$= \sin^2\theta\cdot\sin^2\left(\delta t/\hbar\right), \qquad \delta \equiv (E_+ - E_-)/2 \tag{65}$$

Plotting this expression as a function of time:



These oscillations in the probability of observing the system in the original, unperturbed eigenstates are called "Rabi oscillations."

We can rewrite Eqn. 65 by plugging in our expressions for $E_\pm$ in terms of E_1, E_2, and W from Eqn. 30 to find what is often called "Rabi's formula."

$$\text{Prob}\Big[\,|2\rangle\,\Big] = \frac{4|W|^2}{4|W|^2 + (E_1 - E_2)^2}\,\sin^2\left[\sqrt{4|W|^2 + (E_1 - E_2)^2}\,\frac{t}{2\hbar}\right] \tag{66}$$

What is going on here conceptually? We prepared an initial state $|1\rangle$ that was *not* a stationary state of $\hat{H}$. There is no sloshing of probability the states in the $|+\rangle, |-\rangle$ basis, which are stationary states of $\hat{H}$. But we do see sloshing of state weightings in time in the $|1\rangle, |2\rangle$ basis. The larger the coupling between the states (W), the faster the sloshing.

One way to think about this is that the perturbation *induces* a flopping between the original $|1\rangle$ and $|2\rangle$ states. We can think of $\text{Prob}\big[\,|2\rangle\,\big]$ as the probability that the population of the initially prepared state $|1\rangle$ has been driven into $|2\rangle$ at time t.